Revista de Matemática: Teoría y Aplicaciones 2013 **20**(2) : 155–166
cimpa – ucr issn: 1409-2433

SYMMETRY ORBITS AND THEIR DATA-ANALYTIC PROPERTIES

ÓRBITAS DE SIMETRÍA Y SUS PROPIEDADES EN ANÁLISIS DE DATOS

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Received: 31/Jan/2013; Revised: 17/May/2013; Accepted: 28/May/2013

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Abstract

The concept of data indexed by finite symmetry orbits is reviewed within the data-analytic framework of symmetry studies. Data decompositions are discussed in terms of canonical projections and Plancherel's formulas, and interpreted in terms of orbit invariants.

Keywords: irreducible representations, irreducible characters, finite groups, canonical projections, Fourier transforms, convolution.

Resumen

Se revisa el concepto de datos indexados por órbitas simétricas finitas en el marco de estudios de simetría. Se discute la descomposición de datos en términos de proyecciones canónicas y fórmulas de Plancherel, e interpretada en términos de órbitas invariantes.

Palabras clave: representaciones irreducibles, características irrducibles, grupos finitos, proyecciones canónicas, transformada de Fourier, convolución.

Mathematics Subject Classification: 60B15, 60F05, 05E05.

1 Introduction

Scalar measurements $x : G \longrightarrow \mathbb{R}$ defined on a finite group G can be naturally expressed as points

$$x = \sum_{\tau \in G} x_\tau \tau$$

in the group algebra $\mathbb{C}G$ of G, endowed with an additive (real or complex) vector space structure and a product rule

$$(xy)_\tau = \sum_{\sigma \in G} x_\sigma \; y_{\sigma^{-1}\tau} = (x * y)_\tau$$

given by the convolution $x * y$ of $x, y \in \mathbb{C}G$. A *relabeling* of x by $\sigma \in G$ is given by

$$\sigma x = \sum_{\tau} x_\tau \sigma\tau = \sum_{\gamma} x_{\sigma^{-1}\gamma}\gamma$$

in which the observation x_σ indexed by σ replaces the observation x_1 originally labeled by the identity. The group relabeling gives rise to the usual regular representation

$$\phi_\sigma : (x_\tau)_{\tau \in G} \longmapsto (x_{\sigma^{-1}\tau})_{\tau \in G}$$

of G into $GL_g(\mathbb{C})$, where g indicates the order of G.

The assignment of measurements to objects, although an apparently trivial pursuit,

> ...when measuring some a attribute of a class of objects or events, we associate numbers (or other familiar mathematical entities, such as vectors) with the objects in such a way that the properties of the attributes are faithfully represented as numerical properties[1]...

hides in itself the implicit arbitrariness present in the association or assignment of numbers to objects. As a consequence, it seems desirable that such associations be invariant, in a sense to be made precise later, to any possible relabeling or re-assignment of measurements to labels.

To illustrate the effect of labeling in the assignments of numbers to objects, consider the triangle with vertices $\{(1,0),(0,2),(3,3)\}$ and the evaluation of its area

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

using Heron's Formula, where a, b, and c are the lateral lengths and

$$s = \frac{a+b+c}{2}$$

its semi-perimeter. The evaluation of the area requires the choice of one of the possible permutations

$$\pi : \{a,b,c\} \mapsto \{\sqrt{5}, \sqrt{13}, \sqrt{10}\}$$

giving the distinct indexing of the lateral sides by the symbols $\{a,b,c\}$. In this case, obviously, $\pi A = A$ for all permutations π in the symmetric group S_3, and Heron's Formula is said to reduce *symmetrically.*

In contrast, the evaluation of the triangle's index of handedness, given by

$$F = \frac{a-b}{a+b} + \frac{b-c}{b+c} + \frac{c-a}{c+a},$$

now shows, as one chooses the different permutations π in the labeling of the lateral sides, that the index depends on that particular choice in accordance with

$$\pi F = \begin{cases} -F \text{ for } \pi \in \{(ab),(ac),(bc)\} \subset S_3, \\ +F \text{ for } \pi \in \{1,(abc),(acb)\} \subset S_3 \end{cases}.$$

[1] Foundations of Measurement Volume I: Additive and Polynomial Representations, by David H. Krantz, R. Duncan Luce, Amos Tversky, and Patrick Suppes.

That is, F reduces *anti-symmetrically.*

The purpose of this short communication is presenting a brief overview of the interpretations of orbit relabeling that are relevant to the analysis of measurements indexed along a group orbit. The reader is referred to [1, 2] for complete details and additional references.

2 The canonical projections

Indicate by $\widehat{G}$ the set of irreducible representations of a finite group G, and denote by $\chi_\tau^\xi = Tr\xi_\tau \in \mathbb{C}$ the character of $\xi \in \widehat{G}$ evaluated at $\tau \in G$. When considered as points in $\mathbb{C}G$ we have $\chi^\xi\chi^\eta = 0$ for any distinct non-equivalent $\xi, \eta \in \widehat{G}$, whereas $\chi^\xi\chi^\xi = g\chi^\xi/n_\xi$, with n_ξ denoting the $\mathbb{C}-$dimension of the representation (space of) ξ.

Defining

$$\pi_\xi = \frac{n_\xi}{g}\chi^\xi \in \mathbb{C}G,\ \xi \in \widehat{G},$$

it then follows that

$$\pi_\xi^2 = \pi_\xi; \tag{1}$$

$$\pi_\xi\pi_\eta = 0 \quad \text{for any two distinct } \ \xi, \eta \in \widehat{G}; \tag{2}$$

$$\sum_{\xi\in\widehat{G}} \pi_\xi = 1 \in \mathbb{C}G. \tag{3}$$

This is the abstract group-algebra formulation of the canonical projections theorem.

Given a homomorphism ρ of G into $GL_n(\mathbb{C})$, the linearizations

$$\langle x, \rho\rangle = \sum_\tau x_\tau\rho_\tau, \qquad x \in \mathbb{C}G$$

are points in the enveloping (group) algebra $\mathcal{A}[\rho]$ of $\{\rho_\tau; \tau \in G\}$ such that

$$\langle x, \rho\rangle\langle y, \rho\rangle = \langle xy, \rho\rangle \tag{4}$$

for all $x, y \in \mathbb{C}G$. In particular, for π_ξ as defined above, and ϕ the regular homomorphism of G into $GL_g(\mathbb{C})$, the points

$$\mathcal{P}_\xi = \langle\bar{\pi}_\xi, \phi\rangle = \frac{n_\xi}{g}\sum_\tau \bar{\chi}_\tau^\xi\phi_\tau \in \mathcal{A}[\phi]$$

describe the regular canonical projections and, from (1)-(4), satisfy the fundamental properties

1. $\mathcal{P}_\xi^2 = \mathcal{P}_\xi$;
2. $\mathcal{P}_\xi \mathcal{P}_\eta = \mathcal{P}_\eta \mathcal{P}_\xi = 0$ for any two distinct $\xi, \eta \in \widehat{G}$;
3. $\sum_{\xi \in \widehat{G}} \mathcal{P}_\xi = I$,

where I indicates the $g \times g$ identity matrix. The regular projections give a class of orbit invariants based on the important fact that they commute with all elements ϕ_τ of the regular representation of G. That is

$$\phi_\tau \mathcal{P}_\xi = \mathcal{P}_\xi \phi_\xi$$

for all $\tau \in G$, for all $\xi \in \widehat{G}$.

As a consequence, for all $x \in \mathbb{R}^g$ the relabeling $\phi_\tau(\mathcal{P}_\xi x)$ of $\mathcal{P}_\xi x$ is equal to $\mathcal{P}_\xi(\phi_\tau x)$, which remains in the projection space W_ξ of $\mathcal{P}_\xi$, for all $\tau \in G$.

For example, if $G = S_2 \cong \{1, v\}$, then

$$\phi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \phi_v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so that

$$\mathcal{P}_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is the projection associated with the symmetric character, and

$$\mathcal{P}_\alpha = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

is the projection associated with the anti-symmetric (α). Moreover, $\mathcal{P}_1 x$ defines the subspace W_1 generated by

$$\langle x, 1 \rangle = x_1 + x_v$$

whereas $P_\alpha x$ defines the subspace W_α generated by

$$\langle x, \alpha \rangle = x_1 - x_v.$$

In both cases, $\phi_\tau(\mathcal{P}_\xi x)$ remains in the corresponding (invariant) subspace W_ξ, for $\xi \in \widehat{G} = \{1, \alpha\}$.

This is the notion of orbit relabeling invariance as captured by the regular canonical projections. The two summaries $x_1 + x_v$ and $x_1 - x_v$ give a complete (to be made precise in Section 6) set are invariants for any experiment performed on a bilateral structure such as fellow eyes or follow ears, where the notion of left-right or up-down, may be arbitrary.

The commutativity of ϕ_τ and $\mathcal{P}_\xi$ implies that

$$x'\mathcal{P}_\xi x,\ \xi \in \widehat{G},$$

are orbit *constants*. That is, the relabeling $\tau x = \phi_\tau x$ leads to

$$\begin{aligned}(\phi_\tau x)' P_\xi(\phi_\tau x) &= x'(\phi_\tau' \mathcal{P}_\tau \phi_\tau)x \\ &= x'(\phi_\tau^{-1}\mathcal{P}_\tau \phi_\tau)x \\ &= x'\mathcal{P}_\tau x.\end{aligned}$$

In statistical inference, these invariants are commonly referred to as *variance components* in the decomposition

$$\|x\|^2 = \sum_\xi x'\mathcal{P}_\xi x, \quad \xi \in \widehat{G}$$

of $\|x\|^2$. This is why the canonical projections are the mechanisms behind all decompositions (or analysis) of variance for the purpose of statistically estimating and testing the magnitude of the observed orbit constants (or contrasts, in the statistical terminology).

All random samples have an intrinsic arbitrariness in the assignments of observations to their labels $\{1, 2, \ldots, n\}$, described by the permutations τ in the full symmetric group S_n acting on the labels. If ρ_τ is the corresponding permutation matrix then we write

$$\tau x = \rho_\tau x, \quad \tau \in S_n$$

in analogy to the regular case in which the group acts on itself. The resulting canonical projections give a decomposition

$$I = \mathcal{A} + \mathcal{Q}$$

of the $n \times n$ identity matrix I in which $\mathcal{A}$ is a $n \times n$ matrix with all entries equal to $1/n$ and $\mathcal{Q} = I - \mathcal{A}$. Moreover, $\mathcal{AQ} = \mathcal{QA} = 0$ and $\mathcal{A}^2 = \mathcal{A}$, $\mathcal{Q}^2 = \mathcal{Q}$.

The resulting orbit <u>constants</u> of random sampling are then the components of the decomposition

$$\begin{aligned}\|x\|^2 &= x'\mathcal{A}x + x'\mathcal{Q}x \\ &= n(\bar{x})^2 + \sum_{i=1}^n (x_i - \bar{x})^2,\end{aligned}$$

namely, the sample mean and the sample variance.

Proposition 2.1. *The regular projections evaluate as*

$$\mathcal{P}_\xi x = \frac{n_\xi}{g}\,(\bar{\chi}^\xi * x).$$

PROOF:

$$\begin{aligned}
\mathcal{P}_\xi x &= \frac{n_\xi}{g}\sum_\tau \bar{\chi}^\xi_\tau \phi_\tau x \\
&= \frac{n_\xi}{g}\sum_\tau \bar{\chi}^\xi_\tau \left(\sum_\sigma x_{\tau^{-1}\sigma}\sigma\right) \\
&= \frac{n_\xi}{g}\sum_\sigma \left(\sum_\tau \bar{\chi}^\xi_\tau x_{\tau^{-1}\sigma}\right)\sigma \\
&= \frac{n_\xi}{g}\sum_\sigma \left(\bar{\chi}^\xi * x\right)_\sigma \sigma \\
&= \frac{n_\xi}{g}(\bar{\chi}^\xi * x).
\end{aligned}$$

3 The orbit invariance property

Given $x = \sum_\tau x_\tau \tau \in \mathbb{C}G$ and ρ a homomorphism of G into $GL_n(\mathbb{C})$, we observe that the evaluation of the linearization

$$\langle \sigma x, \rho \rangle$$

of a *relabeled* point

$$\sigma x = \sum_\gamma x_{\sigma^{-1}\gamma}\gamma$$

gives

$$\langle \sigma x, \rho \rangle = \rho_\sigma \langle x, \rho \rangle,$$

so that when $\rho \in \widehat{G}$, the column spaces of $\langle x, \rho \rangle$ are stable representation spaces of ρ and in that sense we say that $\langle x, \xi \rangle$ are orbit invariants of $x \in \mathbb{C}G$, for all $\xi \in \widehat{G}$. These linearizations, for all $\xi \in \widehat{G}$, are called the *Fourier transforms* of x over the finite group G. In other words, The Fourier transforms over a finite group G give precisely a set of orbit invariants for that group.

The data-analytic implication is that the Fourier transforms give precisely a set of data summaries that are orbit invariant. Every instance of experimental results indexed by a (faithful) group orbit can be systematically summarized by evaluating the Fourier transforms over that group.

4 The Fourier–inverse formula

Note that

$$\begin{aligned}
\mathrm{Tr}\ [\xi_{\tau^{-1}}\langle x,\xi\rangle] &= \mathrm{Tr}\ \ \xi_{\tau^{-1}}\sum_{\sigma} x_\sigma \\
&= \sum_{\sigma} x_\sigma\ \mathrm{Tr}\ \ \xi_{\tau^{-1}} = \sum_{\sigma} x_\sigma \chi_{\tau^{-1}\sigma} \\
&= \sum_{\gamma} x_{\tau\gamma}\chi^{\xi}_{\gamma} = \sum_{\gamma} x_{\tau\gamma}\bar{\chi}^{\xi}_{\gamma^{-1}} \\
&= \sum_{\gamma} x_{\tau\gamma^{-1}}\bar{\chi}_{\gamma^{\xi}} = (\bar{\chi}^{\xi} * x)_\tau .
\end{aligned}$$

Therefore, from Proposition 2.1,

$$\begin{aligned}
\sum_{\xi\in\ \widehat{G}} \frac{n_\xi}{g}\ \mathrm{Tr}\ [\xi_{\tau^{-1}}\langle x,\xi\rangle] &= \sum_{\xi} \frac{n_\xi}{g}(\bar{\chi}^{\xi} * x)_\tau \\
&= \left(\sum_{\xi} \mathcal{P}_\xi x\right)_\tau = x_\tau ,
\end{aligned}$$

where in the last equality we used the fact that $\sum_{\xi\in\widehat{G}} \mathcal{P}_\xi = I$. This is then the Fourier–inverse formula:

$$x_\tau = \sum_{\xi\in\widehat{G}} \frac{n_\xi}{g}\mathrm{Tr}\ [\xi_{\tau^{-1}}\langle x,\xi\rangle].$$

The inverse formula is in fact the consequence of the broader result saying that the (algebra) homomorphism

$$x \in \mathbb{C}G \xrightarrow{\varphi} \oplus_\xi \langle x,\xi\rangle \in \prod_{\xi} \mathcal{A}[\xi]$$

is an *isomorphism*. Indeed, if

$$\langle x,\xi\rangle = I_{n_\xi},\ \text{for all}\ \xi\in\widehat{G}$$

then, applying the orbit invariance property

$$\langle \tau x,\xi\rangle = \xi_\tau,\ \text{for all}\ \xi\in\widehat{G}$$

so that

$$\frac{n_\xi}{g}\langle \tau x,\xi\rangle = \frac{n_\xi}{g}\xi_\tau$$

and taking the trace on both sides we have

$$\langle \tau x, \pi_\xi \rangle = \frac{n_\xi}{g}\chi^\xi_\tau = \frac{\chi^1_\tau \chi^\xi_\tau}{g}$$

and summing over $\widehat{G}$ we obtain

$$\langle \tau x, 1 \rangle = \begin{cases} 1 & \tau = 1 \\ 0 & \tau \neq 1 \end{cases}$$

that is, $x = 1$. In the above equality we used the fact the $\sum_{\xi \in \widehat{G}} n_\xi^2 = g$ and that $\sum_{\xi \in \widehat{G}} \pi_\xi = 1$.

5 Fourier basis

In general, it is possible to construct a $g \times g$ unitary matrix $\mathcal{F}$ such that

$$(\mathcal{F}x)_\xi = \sqrt{\frac{n_\xi}{g}} \langle x, \xi \rangle$$

relative to which (basis) we have

$$\mathcal{F}\mathcal{P}_\xi \mathcal{F}^* = \mathrm{diag}(0, \ldots, I_{n_\xi^2}, \ldots, 0)$$

and

$$\mathcal{F}\phi_\tau \mathcal{F}^* = \mathrm{diag}(\ldots, I_{n_\xi} \otimes \xi_\tau, \ldots)_{\xi \in \widehat{G}}.$$

To illustrate consider again the case $G = \{1, v\}$ discussed earlier. Then,

$$\mathcal{F} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathcal{F}\mathcal{F}^* = 1,$$

so that

$$\begin{aligned} \mathcal{F}\mathcal{P}_1\mathcal{F}^* &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ \mathcal{F}\mathcal{P}_2\mathcal{F}^* &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore, in the Fourier basis $\{1/\sqrt{2}, \alpha/\sqrt{2}\}$ the canonical projections are the identity operators in the corresponding (irreducible) subspaces $W_1 \oplus W_\alpha$. This, as the reader may recognize, is just a verification of Schur's Lemma.

Moreover, if $M \in \text{Cent}\{\rho_1, \rho_v\}$ then it follows that M must have the pattern of

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

and, therefore, in the Fourier basis, we have

$$\mathcal{F}M\mathcal{F}' = \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}$$

that is, in W_1

$$M = (a+b)\mathcal{P}_1$$

and in W_α

$$M = (a-b)\mathcal{P}_2$$

or

$$M = (a+b)\mathcal{P}_1 + (a-b)\mathcal{P}_\alpha.$$

Indeed, if $M = f_1\mathcal{P}_1 + f_\alpha\mathcal{P}_\alpha$ then $\mathcal{P}_\xi M = f_\xi \mathcal{P}_\xi$ so that, for $\xi \in \widehat{G}$,

$$f_\xi = \frac{\text{Tr } \mathcal{P}_\xi M}{\text{Tr } \mathcal{P}_\xi} = \frac{\text{Tr } \mathcal{P}_\xi M}{\dim w_\xi} = \begin{cases} \text{Tr } \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix} = (a+b), \xi = 1; \\ \text{Tr } \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix} = (a-b), \xi = \alpha. \end{cases}$$

That is: $\{\mathcal{P}_1, \mathcal{P}_\alpha\}$ is a basis for the center of the (matrix) group $\{\rho_1, \rho_v\}$.

In general, partitioning G into its conjugacy classes $G = [\tau_1] \cup \ldots \cup [\tau_r]$ it follows that

$$\sum_{\eta \in [\tau_i]} \eta = \sum_\sigma \sigma\tau_i\sigma^{-1} \in \text{Cent } \mathbb{C}G,$$

and conversely, if $\sigma\tau\sigma^{-1} = \tau$ for all σ, then $g\sigma = \sum_\sigma \sigma\tau\sigma^{-1} \in \sum_{\eta\in[\tau]} \eta$, so that the sums

$$\sum_{\eta \in [\tau_i]} \eta$$

over the distinct conjugacy classes of G form a basis for the Cent $\mathbb{C}G$.

In addition, note that:

- If $M = f_1\mathcal{P}_1 + f_\alpha\mathcal{P}_\alpha$ then $M^k = f_1^k\mathcal{P}_1 + f_\alpha^k\mathcal{P}_\alpha$, so that $e^{-iMt} = e^{-if_1t}\mathcal{P}_1 + e^{-if_\alpha t}\mathcal{P}_\alpha$ describes the superpositions of wave-like functions.

- If $\mathcal{F}$ indicates the Fourier basis introduced above, and X is a random variable with distribution $\mathcal{N}(\mu, \Sigma)$ with Σ in Cent $\mathbb{C}G$, then the distribution of $\mathcal{F}X$ is $\mathcal{N}(\mathcal{F}\mu, \mathcal{F}\Sigma\mathcal{F}^*)$, with

$$\mathcal{F}\Sigma\mathcal{F}^* = \begin{bmatrix} a+b & \\ & a-b \end{bmatrix}, \quad \mathcal{F}\mu = \begin{pmatrix} \mu_1 + \mu_2 \\ \mu_1 - \mu_2 \end{pmatrix}$$

  and

$$\Sigma = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}, \quad \begin{array}{l} a+b = \sigma^2(1+\rho) \\ a-b = \sigma^2(1-\rho) \end{array}.$$

- If $\Sigma x = \lambda x$ and $\Sigma \in$ Cent $\mathbb{C}G$, then $\Sigma(\mathcal{P}_\xi x) = \mathcal{P}_\xi(\Sigma x) = \mathcal{P}_\xi(\lambda x) = \lambda(\mathcal{P}_\xi x)$, that is, all $\mathcal{P}_\xi x$ are eigenvectors associated with the eigenvalue λ.

- Symmetric tensors. Define $\Pi_f(A) = \underbrace{A \otimes \ldots \otimes A}_{f \text{ copies}}$. Take $M = \lambda_1\mathcal{P}_1 + \lambda_\alpha\mathcal{P}_\alpha \in$ Cent $\mathbb{C}G$. Then $\Pi_f(M)$ is a symmetric tensor [3] in the sense that it commutes with $\Pi_f(\rho_\tau)$ for all $\tau \in G$. That is, the commutator

$$[\Pi_f(\rho_\tau), \Pi_f(M)] = 0, \forall\, \tau \in G.$$

  In quantum mechanics, the symmetric tensors are the observables of the system, whereas

$$\Pi_f(M)x$$

  are the possible states of the system.

To illustrate, let $f = 2$, so that

$$\begin{aligned} \Pi_2(M) \quad = \quad & \lambda_1^2\mathcal{P}_1 \otimes \mathcal{P}_1 + \lambda_1\lambda_2\mathcal{P}_1 \otimes \mathcal{P}_\alpha \\ & + \lambda_2\lambda_1\mathcal{P}_\alpha \otimes \mathcal{P}_1 + \lambda_1^2\mathcal{P}_\alpha \otimes \mathcal{P}_\alpha \end{aligned}$$

is a symmetric tensor. Take any x as a superposition of

$$\{\chi_1 \otimes \chi_1, \chi_1 \otimes \chi_\alpha, \chi_\alpha \otimes \chi_1, \chi_\alpha \otimes \chi_\alpha\}.$$

Then,

$$\begin{aligned}
\Pi_2(M)(\chi_1 \otimes \chi_1) &= \lambda_1^2\, \chi_1 \otimes \chi_2, \\
\Pi_2(M)(\chi_1 \otimes \chi_\alpha) &= \lambda_1\lambda_2 \chi_1 \otimes \chi_\alpha, \\
\Pi_2(M)(\chi_\alpha \otimes \chi_1 &= \lambda_2\lambda_1 \chi_\alpha \otimes \chi_1, \\
\Pi_2(M)(\chi_\alpha \otimes \chi_\alpha) &= \lambda_2^2 \chi_\alpha \otimes \chi_\alpha.
\end{aligned}$$

Note that the two states $\chi_1 \otimes \chi_\alpha$ and $\chi_\alpha \otimes \chi_1$ share the same (energy) level $\lambda_1\lambda_2$.

6 Parseval's and Plancherel's equalities

From the Fourier basis definition it follows that

$$\|x\|^2 = x^* x = (\mathcal{F}x)^*(\mathcal{F}x) = \sum_{\xi \in \widehat{G}} \frac{n_\xi}{g} \|\langle x, \xi \rangle\|^2,$$

which is the Parseval's equality. Similarly we obtain Plancherel's equality

$$x^* y = \sum_{\xi \in \widehat{G}} \frac{n_\xi}{g} \|\langle x, \xi \rangle\| \; \|\langle y, \xi \rangle\|.$$

Acknowledgments

This short review is based on the partial content of a short course offered at the University of Costa Rica in 2012. The author is thankful to all colleagues at CIMPA for the opportunity of a friendly and productive visit. The author is also thankful for their assistance in the preparation of these notes.

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