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A NEW METHOD FOR THE ANALYSIS OF IMAGES: THE SQUARE WAVE METHOD

UN NUEVO MÉTODO PARA EL ANÁLISIS DE IMÁGENES: EL MÉTODO DE LAS ONDAS CUADRADAS

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Abstract

The Square Wave Method (SWM) – previously applied to the analysis of signals – has been generalized here, quite naturally and directly, for the analysis of images. Each image to be analyzed is subjected to a process of digitization so that it can be considered to be made up of pixels. A numeric value or "level" ranging from 0 to 255 (on a gray scale going from black to white) corresponds to each pixel. The analysis process described causes each image analyzed to be "decomposed" into a set of "components". Each component consists of a certain train of square waves. The SWM makes it possible to determine these trains of square waves unambiguously. Each row and each column of the image analyzed can be obtained once again by adding all the trains of square waves corresponding to a particular row or to a particular column. In this article the entities analyzed were actually sub-images of a certain digitized image. Given that any sub-image of any image is also an image, it was feasible to apply the SWM for the analysis of all the sub-images.

Keywords: analysis of images, square wave method

Resumen

El método de las ondas cuadradas —previamente aplicado al análisis de señales— es generalizado, de manera directa y natural, para el análisis de imágenes. Cada imagen a ser analizada es sometida, en primer lugar, a un proceso de digitalización que posibilita considerarla constituida por pixels. A cada uno de estos pixels le corresponde un valor numérico o "nivel"—desde 0 hasta 255— en una escala de grises que se extiende desde el negro al blanco. El proceso de análisis presentado conduce a la "descomposición" de cada imagen analizada en un conjunto de "componentes". Cada componente consiste en cierto tren de ondas cuadradas. El método de las ondas cuadradas permite determinar, sin ambigüedad, cuáles son dichos trenes de ondas cuadradas. Cada fila, así como cada columna, de la imagen analizada puede ser reobtenida sumando todos los trenes de ondas cuadradas correspondientes a dicha fila o a dicha columna. En este trabajo, los entes sometidos a análisis fueron, en realidad, subimágenes de una cierta imagen digitalizada. Dado que cualquier subimagen de cualquier imagen es, también, una imagen, resultó factible aplicar el método de las ondas cuadradas para el análisis de todas las subimágenes.

Palabras clave: análisis de imágenes, método de las ondas cuadradas

Mathematics Subject Classification: 68U10, 94A12.

1 Introduction

Previously, consideration was given to the analysis of functions of one variable with the Square Wave Method (SWM) [5]. This method, which will be reviewed briefly in the following section, can be generalized for functions of two variables. This generalization makes it possible to apply the SWM to the analysis of images.

The objective of this article is to specify how the SWM can be applied to the analysis of images.

2 Brief review of the application of the SWM to the analysis of functions of one variable

Let $f(x)$ be a function of one variable, which in a given interval Δx, satisfies the conditions of Dirichlet [3]. Then, that function can be approximated in that interval by means of a certain sum of trains of square waves. The use of the SWM makes it possible to specify these trains of square waves unambiguously.

Consider, for example, the function $f(x)$, as indicated below:

$$f(x) = \frac{\ln x \cdot e^x}{x^3}; \quad 1 \leq x \leq 4 \tag{1}$$

In figure 1, $f(x)$ is shown for the interval specified in (1).

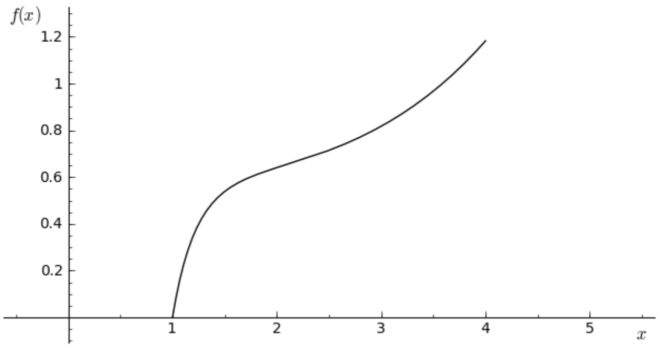


Figure 1: $f(x) = \frac{\ln x \cdot e^x}{x^3}$, $1 \leq x \leq 4$.

Note that the interval of x (Δx) in which $f(x)$ will be analyzed has a length of 3 units: $\Delta x = 4 - 1 = 3$.

First, an explanation will be given about how to proceed if one wants to obtain an approximation to $f(x)$, in the interval Δx specified in (1), made up of the sum of 6 trains of square waves. The interval Δx is then divided into a number of sub-intervals – of equal length – which is the same as the number of the trains of square waves. In this case, there will be 6 sub-intervals. The approximation to $f(x)$ to be obtained in the interval Δx will be the sum of 6 trains of square waves: S_1, S_2, S_3, S_4, S_5, and S_6. The first of the trains of square waves is denominated S_1, the second S_2, and so on.

Each of these trains of waves S_i, for $i = 1, 2, 3, 4, 5$ and 6, will be characterized by a certain "spatial frequency" f_i; that is, the number of waves in the train of square waves considered which is contained in the unit of length, and a certain amplitude.

For the case considered here, a description will be provided below of how the amplitudes corresponding to the different trains of square waves are determined (see figure 2).

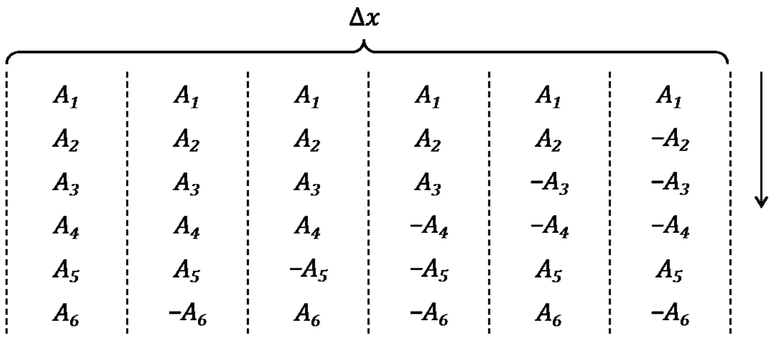


Figure 2: How to apply the SWM to the analysis of the function represented in figure 1 (see indications in text).

The vertical arrow pointing down at the right of figure 2 indicates how to add the terms corresponding to each one of the 6 sub-intervals of Δx. Thus, to obtain the values of the amplitudes corresponding to S_1, S_2, S_3, S_4, S_5 and S_6, the following system of linear algebraic equations must be solved:

$$\left.\begin{array}{l}A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = V_1 \\ A_1 + A_2 + A_3 + A_4 + A_5 - A_6 = V_2 \\ A_1 + A_2 + A_3 + A_4 - A_5 + A_6 = V_3 \\ A_1 + A_2 + A_3 - A_4 - A_5 - A_6 = V_4 \\ A_1 + A_2 - A_3 - A_4 + A_5 + A_6 = V_5 \\ A_1 - A_2 - A_3 - A_4 + A_5 - A_6 = V_6.\end{array}\right\} \quad (2)$$

In the preceding system of linear algebraic equations (2), V_1, V_2, V_3, V_4, V_5, and V_6 are the values for $f(x)$ as specified in (1) at the mid-points of the first, second, third, fourth, fifth, and sixth sub-intervals, respectively, of the interval Δx in which $f(x)$ is analyzed. It follows that the values V_i, for $i = 1, 2, 3, 4, 5$ and 6, can be computed given that $f(x)$ has been specified in (1). These values are: $V_1 = 0.398769931690279$, $V_2 = 0.600884713383228$, $V_3 = 0.675458253887002$, $V_4 = 0.760888387081983$, $V_5 = 0.885510437891641$, and $V_6 = 1.06576570701653$.

The six unknowns of the system of equations specified in (2) are A_1, A_2, A_3, A_4, A_5, and A_6. $|A_i|$ refers to the "amplitude" of the train of square waves S_i for $i = 1, 2, \ldots, 6$. The (constant) value of each positive square semi-wave of the train of square waves S_i is $|A_i|$, and the (constant) value of each negative square semi-wave of that S_i is $-|A_i|$. For the case discussed here, in which Δx was divided into 6 equal sub-intervals, the spatial frequencies corresponding to the different trains of waves are the following:

$$f_1 = \frac{1}{2\Delta x}\left(\frac{6}{6-1+1}\right) = \frac{1}{2\Delta x}\left(\frac{6}{6}\right) = \frac{1}{6}$$
$$f_2 = \frac{1}{2\Delta x}\left(\frac{6}{6-2+1}\right) = \frac{1}{2\Delta x}\left(\frac{6}{5}\right) = \frac{1}{5}$$
$$f_3 = \frac{1}{2\Delta x}\left(\frac{6}{6-3+1}\right) = \frac{1}{2\Delta x}\left(\frac{6}{4}\right) = \frac{1}{4}$$
$$f_4 = \frac{1}{2\Delta x}\left(\frac{6}{6-4+1}\right) = \frac{1}{2\Delta x}\left(\frac{6}{3}\right) = \frac{1}{3}$$
$$f_5 = \frac{1}{2\Delta x}\left(\frac{6}{6-5+1}\right) = \frac{1}{2\Delta x}\left(\frac{6}{2}\right) = \frac{1}{2}$$
$$f_6 = \frac{1}{2\Delta x}\left(\frac{6}{6-6+1}\right) = \frac{1}{2\Delta x}\left(\frac{6}{1}\right) = \frac{1}{1} = 1.$$

In general, if the spatial interval Δx – whose length, of course, may be different from 3 – is divided into n equal sub-intervals, the spatial frequencies corresponding to the different n trains of square waves are as follows:

$$f_i = \frac{1}{2\Delta x}\left(\frac{n}{n-i+1}\right); \quad i = 1, 2, \ldots, n. \tag{3}$$

It can be useful to take interval Δx as a unit of length. Therefore, the values obtained for the spatial frequencies of the diverse trains of square waves are:

$$\begin{aligned} f_1 &= \frac{1}{2}\left(\frac{6}{6}\right) = \frac{1}{2} & f_2 &= \frac{1}{2}\left(\frac{6}{5}\right) = \frac{3}{5} \\ f_3 &= \frac{1}{2}\left(\frac{6}{4}\right) = \frac{3}{4} & f_4 &= \frac{1}{2}\left(\frac{6}{3}\right) = 1 \\ f_5 &= \frac{1}{2}\left(\frac{6}{2}\right) = \frac{3}{2} & f_6 &= \frac{1}{2}\left(\frac{6}{1}\right) = 3. \end{aligned}$$

The accuracy of the preceding equations may be checked by consulting figure 2. Thus, half of the square wave S_1, for example, occupies the new unit of length Δx, for $f_1 = \frac{1}{2}$; one square wave of S_4 occupies Δx, for $f_4 = 1$; three square waves of S_6 occupy Δx, for $f_6 = 3$; etc.

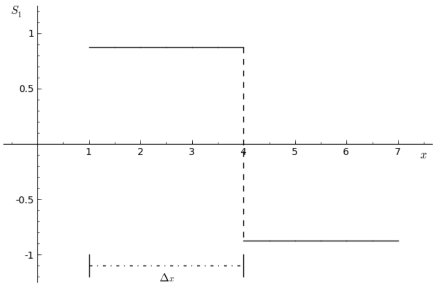
If Δx is taken as the unit of length, and if it is admitted that Δx is divided into n sub-intervals, the general expression for the frequencies of trains of square waves $S_1, S_2, \ldots, S_n$ is as follows:

$$f_i = \frac{1}{2}\left(\frac{n}{n-i+1}\right); \quad i = 1, 2, \ldots, n. \tag{4}$$

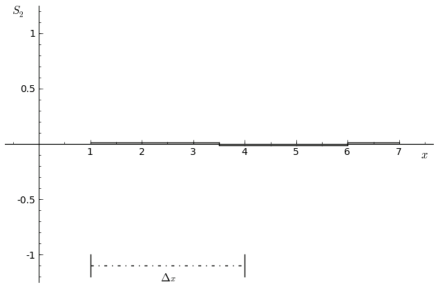
The system of equations (2) was solved and the following results obtained for the unknowns:

$$\begin{aligned} A_1 &= 0.870611980451768; & A_2 &= 0.0109297562840276; \\ A_3 &= -0.301712577349665; & A_4 &= 0.0583423242489839; \\ A_5 &= -0.138344161098361; & A_6 &= -0.101057390846474. \end{aligned}$$

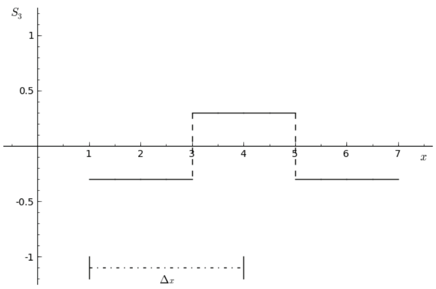
The trains of square waves S_1, S_2, S_3, S_4, S_5, and S_6 have been represented in figures 3a, 3b, 3c, 3d, 3e, and 3f, respectively.



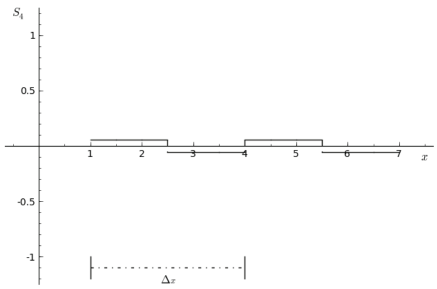
(a) $S1(x)$.



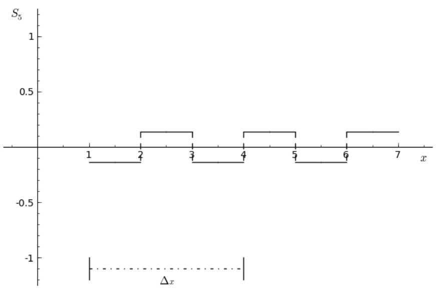
(b) $S2(x)$.



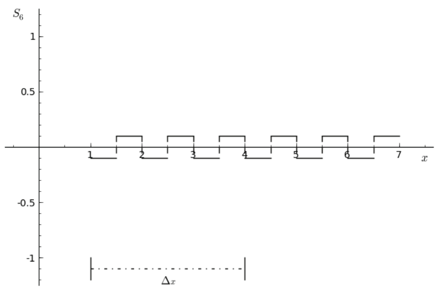
(c) $S3(x)$.



(d) $S4(x)$.



(e) $S5(x)$.



(f) $S6(x)$.

Figure 3: Trains of square waves S_1, S_2, S_3, S_4, S_5, and S_6.

The approximation obtained for $f(x)$ (as specified in (1), in interval Δx, by adding the six trains of square waves) has been displayed in figure 4.

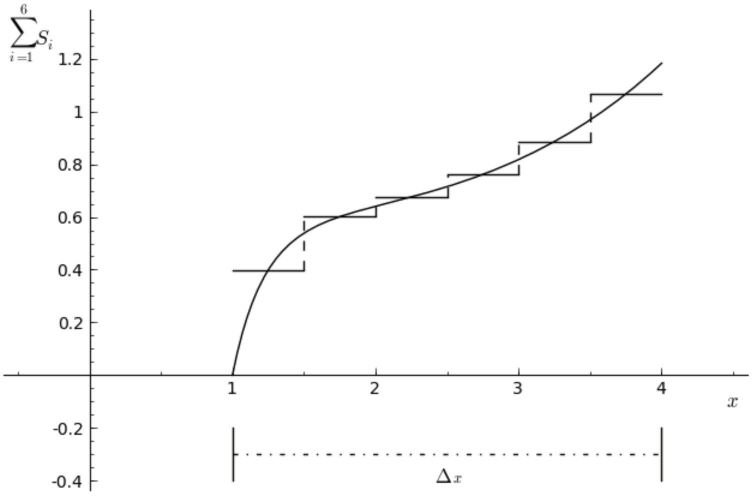


Figure 4: Approximation to $f(x)$, specified in (1) by $\sum_{i=1}^{6} S_i$; $f(x)$ is indicated by a solid line.

If one wants to achieve, in interval Δx, by adding the trains of square waves, a better approximation to $f(x)$, then Δx should be divided into a larger number of equal sub-intervals. The larger the number of these sub-intervals, the better the approximation. Thus, for example, the approximation to $f(x)$ that can be achieved if Δx is divided into 18 sub-intervals of equal length, is shown in figure 5. In this case, it is clear that 18 trains of square waves were added up.

The SWM cannot be considered a branch of Fourier Analysis; that is, the trains of square waves S_i, for $i = 1, 2, \ldots, n$, do not make up a system of orthogonal functions.

3 Characterization of the application of the SWM to the analysis of images

First of all, the images to be analyzed must be digitized. It will be supposed that the digitized images are made up of a set of pixels, and that for each pixel a certain level of gray may be established on the gray scale,

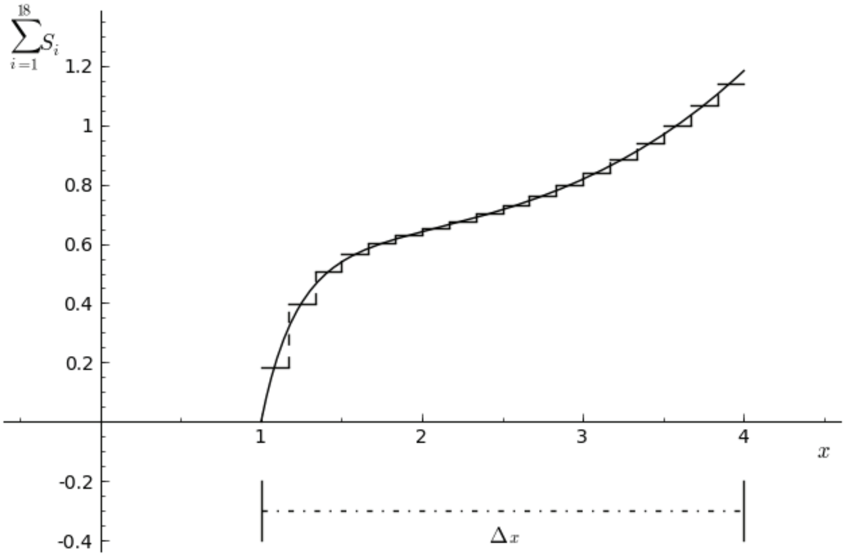


Figure 5: Approximation to $f(x)$ specified in (1) by $\sum_{i=1}^{18} S_i$; $f(x)$ is indicated by a solid line.

ranging from black (0) to white (255). The possible numerical values in the gray scale are as follows: $0, 1, 2, \ldots, 255$. In other words, these numerical values range from 0 (black) up to 255 (white), one unit at a time.

It may be admitted that the image to be analyzed occupies a rectangular region, as usually occurs with pictures or photographs. (If the sides of that rectangle are the same, as is acceptable, the rectangle is a square.)

To specify how the SWM is applied to the analysis of images, it will be supposed initially that the region to be analyzed is made up of a small number of pixels. (Later, this restriction will be eliminated.) Let there be a rectangle such as that shown in figure 6.

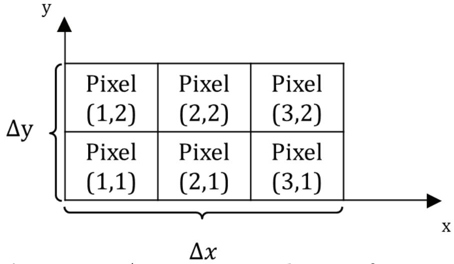


Figure 6: An image made up of 6 pixels.

This rectangle is composed of two rows, each made up of 3 pixels. Likewise, it may be considered to be composed of three columns, each of 2 pixels. Of course, the total number of pixels in the rectangle is 6.

Let it be admitted that the image to be analyzed is located in the area $\Delta x \cdot \Delta y$. The interval Δx has been divided into 3 equal sub-intervals. The interval Δy has been divided into 2 equal sub-intervals.

In figure 7, a set of symbols has been added to the rectangle in figure 6, both for the x-axis and for the y-axis. These symbols are of the type shown in figure 2 (where the y-axis was not considered; therefore, symbols such as B_1, B_2, and $-B_2$ were unnecessary).

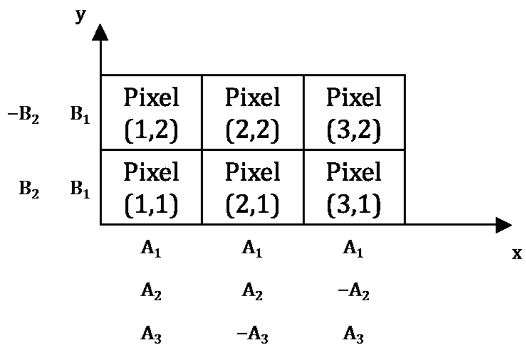


Figure 7: How to apply the SWM to the analysis of the image corresponding to figure 6.

In figures 6 and 7, the pixels have been named conventionally as might be expected, given the system of orthogonal Cartesian coordinates used for reference.

For each of the pixels in the rectangle considered, there is a linear algebraic equation.

A description will be provided below of how to find the linear algebraic equation corresponding to pixel $(3, 2)$. The same approach is used to obtain the equations corresponding to each of the other 5 pixels.

Note that the third sub-interval of Δx and the second sub-interval of Δy correspond to pixel $(3, 2)$. The symbols associated with the third sub-interval of Δx are: $A_1, -A_2$, and A_3). The symbols associated with the second sub-interval of Δy are: B_1, and $-B_2$). Consideration will be given first to the Cartesian product of the following sets: $(A_1, -A_2, A_3)$ and $(B_1, -B_2)$:

$$
\begin{aligned}
&(A_1, -A_2, A_3) \times (B_1, -B_2) = \\
&\quad = \ ((A_1, B_1), (A_1, -B_2), (-A_2, B_1), (-A_2, -B_2), (A_3, B_1), (A_3, -B_2)) \, .
\end{aligned}
$$

An operator O is introduced, which operates on each of the elements belonging to that Cartesian product. (Each one of these elements is an ordered pair.) When that operator acts on an ordered pair, a term of a certain algebraic sum (to be specified) is obtained, to which a positive sign must be attributed if the elements of the ordered pair have the same signs, and a negative sign if the elements of the ordered pair have different signs. The term obtained has two subscripts such that the first of them is the same as the subscript of the first symbol of the ordered pair and the second subscript is the same as the subscript of the second symbol of the ordered pair.

According to the explanation provided in the above paragraph, the following results are obtained:

$$\begin{aligned} O(A_1, B_1) &= C_{1,1} \\ O(A_1, -B_2) &= -C_{1,2} \\ O(-A_2, B_1) &= -C_{2,1} \\ O(-A_2, -B_2) &= C_{2,2} \\ O(A_3, B_1) &= C_{3,1} \\ O(A_3, -B_2) &= -C_{3,2}. \end{aligned} \tag{5}$$

The right-hand members of the above equalities are the terms mentioned. If the algebraic sum of those terms is made equal to the numerical value corresponding to the level of gray of pixel $(3, 2)$, $V_{3,2}$, the following linear algebraic equation is obtained for that pixel:

$$C_{1,1} - C_{1,2} - C_{2,1} + C_{2,2} + C_{3,1} - C_{3,2} = V_{3,2}.$$

If the same procedure is applied for each of the other 5 pixels of the rectangle, 5 more linear algebraic equations are obtained. Together with the first one, they make up the following system of linear algebraic equations:

$$\begin{aligned} C_{1,1} + C_{1,2} + C_{2,1} + C_{2,2} + C_{3,1} + C_{3,2} &= V_{1,1} \\ C_{1,1} - C_{1,2} + C_{2,1} - C_{2,2} + C_{3,1} - C_{3,2} &= V_{1,2} \\ C_{1,1} + C_{1,2} + C_{2,1} + C_{2,2} - C_{3,1} - C_{3,2} &= V_{2,1} \\ C_{1,1} - C_{1,2} + C_{2,1} - C_{2,2} - C_{3,1} + C_{3,2} &= V_{2,2} \\ C_{1,1} + C_{1,2} - C_{2,1} - C_{2,2} + C_{3,1} + C_{3,2} &= V_{3,1} \\ C_{1,1} - C_{1,2} - C_{2,1} + C_{2,2} + C_{3,1} - C_{3,2} &= V_{3,2}. \end{aligned} \tag{6}$$

It is admitted that the values $V_{i,j}$ (for $i = 1, 2, 3$; and $j = 1, 2$) are known.

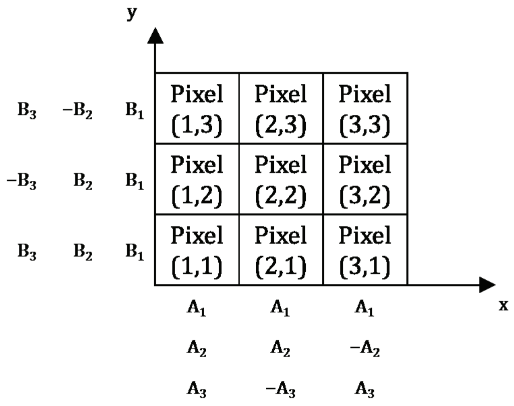


Figure 8: How to apply the SWM to the analysis of the image represented here, made up of 9 pixels.

Another example similar to that of figure 7 has been provided in figure 8 to show the system of linear algebraic equations obtained.

The system of linear algebraic equations obtained by applying the SWM to the image in figure 8 is as follows:

$$\begin{aligned}
C_{1,1}+C_{1,2}+C_{1,3}+C_{2,1}+C_{2,2}+C_{2,3}+C_{3,1}+C_{3,2}+C_{3,3}&=V_{1,1}\\
C_{1,1}+C_{1,2}-C_{1,3}+C_{2,1}+C_{2,2}-C_{2,3}+C_{3,1}+C_{3,2}-C_{3,3}&=V_{1,2}\\
C_{1,1}-C_{1,2}+C_{1,3}+C_{2,1}-C_{2,2}+C_{2,3}+C_{3,1}-C_{3,2}+C_{3,3}&=V_{1,3}\\
C_{1,1}+C_{1,2}+C_{1,3}+C_{2,1}+C_{2,2}+C_{2,3}-C_{3,1}-C_{3,2}-C_{3,3}&=V_{2,1}\\
C_{1,1}+C_{1,2}-C_{1,3}+C_{2,1}+C_{2,2}-C_{2,3}-C_{3,1}-C_{3,2}+C_{3,3}&=V_{2,2}\\
C_{1,1}-C_{1,2}+C_{1,3}+C_{2,1}-C_{2,2}+C_{2,3}-C_{3,1}+C_{3,2}-C_{3,3}&=V_{2,3}\\
C_{1,1}+C_{1,2}+C_{1,3}-C_{2,1}-C_{2,2}-C_{2,3}+C_{3,1}+C_{3,2}+C_{3,3}&=V_{3,1}\\
C_{1,1}+C_{1,2}-C_{1,3}-C_{2,1}-C_{2,2}+C_{2,3}+C_{3,1}+C_{3,2}-C_{3,3}&=V_{3,2}\\
C_{1,1}-C_{1,2}+C_{1,3}-C_{2,1}+C_{2,2}-C_{2,3}+C_{3,1}-C_{3,2}+C_{3,3}&=V_{3,3}.
\end{aligned} \tag{7}$$

4 Application of the SWM to an image – a digitized photograph of the Taj Mahal – composed of 786,432 Pixels

The digitized image to be analyzed is displayed in figure 9.

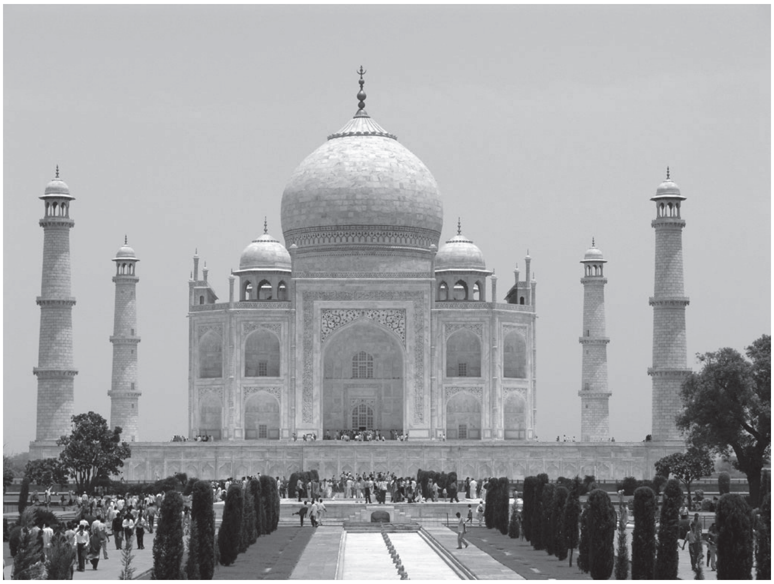


Figure 9: Image of the Taj Mahal.

The image in figure 9 is made up of 786,432 pixels. Each row of the image contains 1,024 pixels and each column 768 pixels.

If the SWM were applied directly to the analysis of the image in the figure, a system of 786,432 linear algebraic equations with that same number of unknowns would have to be solved. That task would exceed the operational capacity of the computer used for this purpose. Therefore, it was necessary to divide the image into 768 sections. Each of these parts is a square made up of 1,024 pixels. In other words, each square sub-image obtained by dividing the image in figure 9, may be considered to be made up of 32 rows, each with 32 pixels (or 32 columns, each with 32 pixels also). To conduct the analysis using the SWM for every sub-image means solving a system of 1,024 linear algebraic equations, and that was feasible using the technical resources available.

Using the SWM, the analysis was carried out for all of the sub-images mentioned above. Thus, for each sub-image, it was possible to obtain

1,024 terms: $C_{i,j}; i = 1, 2, 3, \ldots, 32; j = 1, 2, 3, \ldots, 32$. Once those terms are computed, it is feasible to obtain, for each pixel of each sub-image, the corresponding algebraic sum whose value is made equal to the level of gray of that pixel.

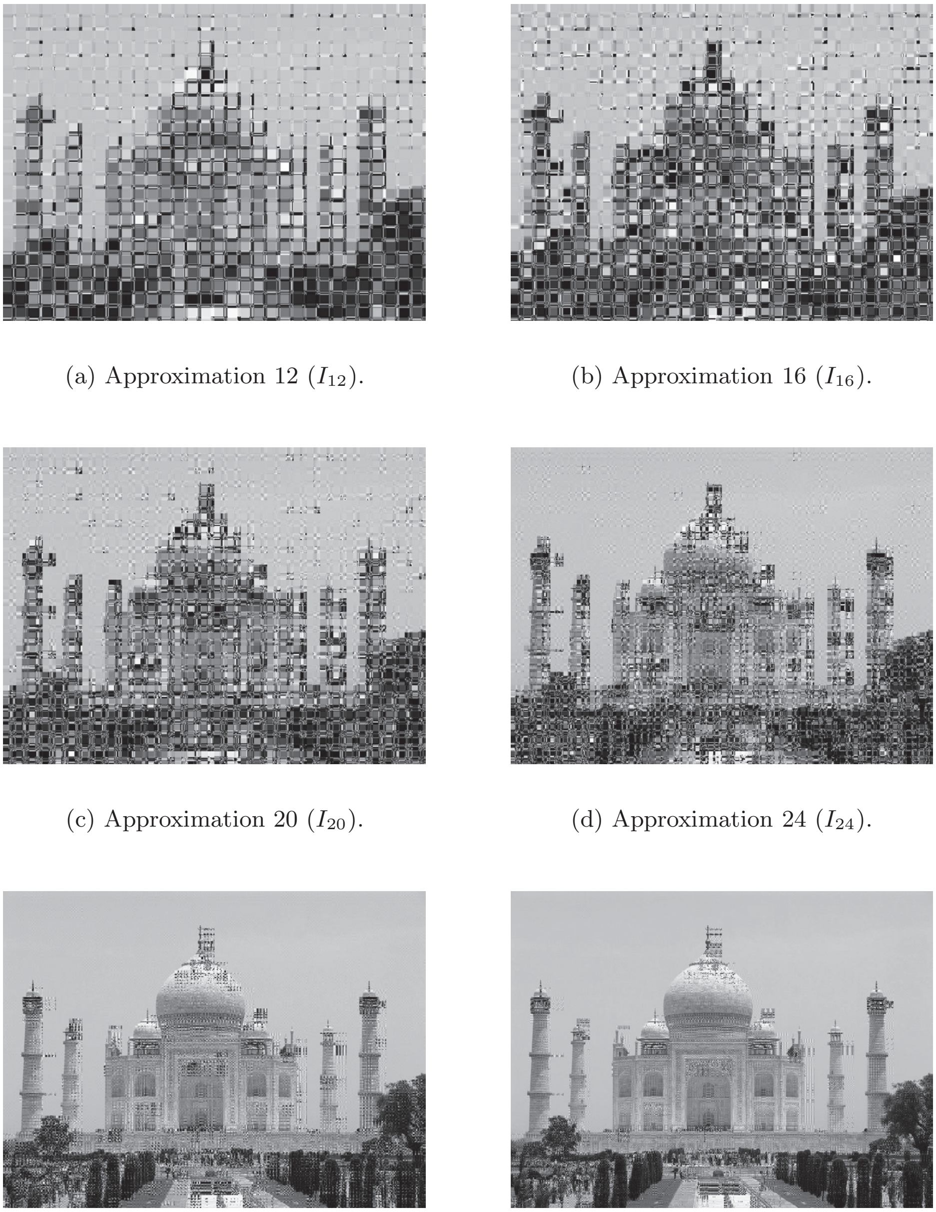
Given the characteristics provided for the image (in figure 9) to be analyzed and the sub-images into which it was divided, it may be considered that this image is made up of 24 rows of sub-images, or 32 columns of sub-images. Each sub-image belongs to only one of the 24 rows and to only one of the 32 columns.

What is referred to here as "approximation $(m, n) - I_{m,n}$" ($1 \leq m \leq 32$ and $1 \leq n \leq 32$), to the digitized image analyzed will be specified below. Suppose, for example, that one wants to obtain $I_{7,24}$ for the image analyzed. For this purpose, the sum is obtained algebraically for each pixel of all of the sub-images, only for those terms $C_{i,j}$ such that $1 \leq i \leq 7$ and $1 \leq j \leq 24$. Therefore, the terms, $C_{5,11}$, $C_{3,24}$, and $C_{4,2}$, for instance, would be terms of the algebraic sum corresponding to each pixel. However, terms such as $C_{5,28}$ (note that $28 > 24$), $C_{27,32}$ (note that $27 > 7$ and $32 > 24$), and $C_{9,24}$ (note that $9 > 7$) are not included in the algebraic sum corresponding to each pixel.

It will be admitted that when both subscripts are the same for an approximation of the type discussed (such as $I_{17,17}$), one subscript suffices; in this case, I_{17}. With this convention, it is evident that I_{32} is the same as the image analyzed. (Hence, that image is one of the sub-images mentioned above.)

From that image, I_{12}, I_{16}, I_{120}, I_{24}, I_{28}, I_{29}, I_{30}, I_{31}, and I_{32} are displayed in figures 10a, 10b, 10c, 10d, 10e, 10f, 10g, 10h, and 10i, respectively. Note that I_{32} coincides with that image, as shown in figure 9. (Of course, when stating that any I_i, for $i = 1, 2, \ldots, 32$, is obtained for the image in figure 9, what is meant is that I_i is obtained for each of the 768 sub-images into which the image was divided.)

Reference may be made to each sub-image in the following way: Suppose that the rows of the sub-images are numbered from bottom up, from 1 to 24, and that the columns of the sub-images are numbered from left to right, from 1 to 32. Given that each sub-image belongs to only one row of sub-images and to only one column of sub-images, each sub-image can be unambiguously specified by indicating the row and column to which it belongs. Thus, for example, sub-image (8, 27) is that belonging to column 8 and row 27, according to the convention that the first element of the ordered pair (8, 27) refers to the column and the second to the row.



(a) Approximation 12 (I_{12}). (b) Approximation 16 (I_{16}).

(c) Approximation 20 (I_{20}). (d) Approximation 24 (I_{24}).

(e) Approximation 28 (I_{28}). (f) Approximation 29 (I_{29}).

Figure 10: Different approximations to the image in figure 9.

(g) Approximation 30 (I_{30}).

(h) Approximation 31 (I_{31}).

(i) Approximation 32 (I_{32}).

Figure 10: Different approximations to the image in figure 9 (Cont.).

The contribution of each term $C_{i,j}$, for $i = 1, 2, \ldots, 32$; and $j = 1, 2, \ldots, 32$, to the value of the level of gray of each pixel in each row and in each column of each sub-image can be positive, negative, or zero. If for any sub-image of those considered, the contribution of any term to the value of the level of gray of each pixel in any row or column is graphed, what is obtained is the representation of part of a train of square waves. To specify each train of square waves, a notation will be introduced and it will be illustrated by two examples.

Suppose that one wants to graph the contribution of $C_{7,24}$ to each pixel in row 18 of sub-image (15, 23). Reference will be made to the train of square waves to which the graph belongs, in the following way:

$$S\left((15, 23), C_{7,24}, (\quad, 18)\right).$$

It was shown that the train of square waves considered is a function of an ordered set of three elements. The first element is the ordered pair, (15,23), which specifies the sub-image to which that particular row belongs. The second element indicates the term considered: $C_{7,24}$. The third element $(\ \ ,18)$ is a "pseudo-ordered-pair", whose first element is missing, and whose second element refers to row 18.

Here is another example: $S((7,10), C_{15,30}, (12,\ \ ))$ is the train of square waves representing the contribution of element $C_{15,30}$ to each pixel in column 12 of the sub-image (7, 10). In figures 11a and 11b, this has been graphed in trains of square waves.

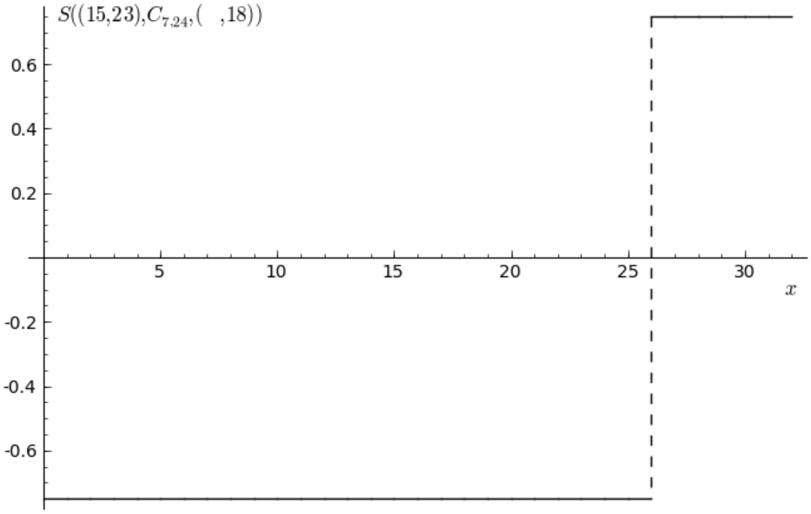
To determine the spatial frequency of any train of square waves of the type represented in figure 11a and 11b, how each term $C_{i,j}$ was obtained should be kept in mind (see (5)). If the contribution of that term to a row is taken into account, the first subscript of $C_{i,j}$ must be considered and formula (3) applied (or (4), according to which length is chosen as the unit of length: the length of a side of a pixel or the length of Δx, respectively).

Suppose that one wishes to determine the spatial frequency of $S((15,23), C_{7,24}, (\ \ ,18))$ using first the length of one side of a pixel (which may be any side, since it is accepted that pixels are square) as the unit of length. Given that the train of square waves considered corresponds to the contribution of the term $C_{7,24}$ to a row (row 18), the first subscript (7) of that term must be taken into account. Given that in this case the image analyzed is one of the sub-images used (one square such that $\Delta x = \Delta y = 32$), upon applying (3), the spatial frequency for that train of square waves is obtained:

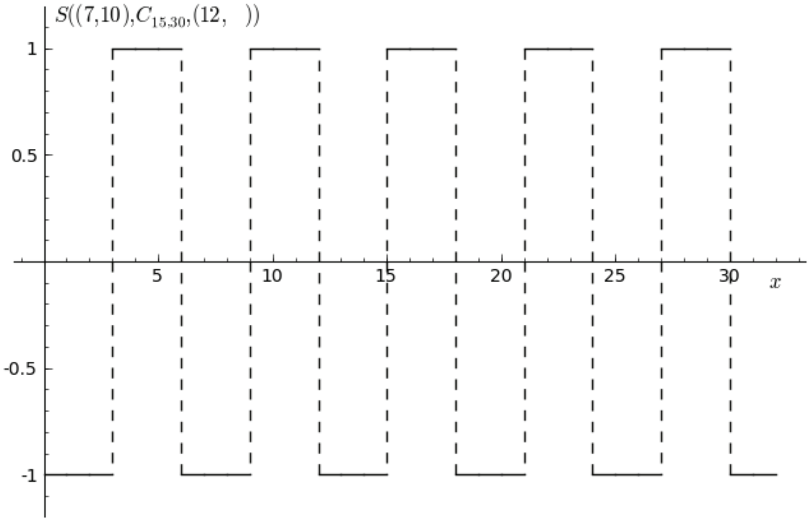
$$f_7 = \frac{1}{2\Delta x}\left(\frac{32}{32-7+1}\right) = \frac{1}{2}\left(\frac{1}{26}\right) = \frac{1}{52}.$$

If, on the other hand, Δx is the unit of length used to calculate the corresponding frequency, by applying (4), the following result is obtained:

$$f_7 = \frac{1}{2}\left(\frac{32}{26}\right) = \frac{16}{26} = \frac{8}{13}.$$



(a) $[S((15,23),C_{7,24},(\quad ,18))$.



(b) $S((7,10),C_{15,30},(12, \quad))$.

Figure 11: Trains of square waves specified in (a) and (b).

Suppose that one wishes to determine the spatial frequency of $S((7,10), C_{15,30}, (12, \quad))$ using, first of all, the length of one side of a pixel as a unit of length. Given that the train of square waves considered corresponds to the contribution of term $C_{15,30}$ to a column (column 12), the second subscript (30) of that term should be taken into account. By (3), the following result is obtained:

$$f_{30} = \frac{1}{2\Delta x}\left(\frac{32}{32-30+1}\right) = \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{6}.$$

If, however, Δy is chosen as the unit of length, and it is recalled that for the case considered, $\Delta x = \Delta y = 32$, by applying (4), the following result is obtained:

$$f_{30} = \frac{1}{2}\left(\frac{32}{32-30+1}\right) = \frac{16}{3}.$$

5 Discussion and perspectives

5.1 Discussion

Analyzing any entity implies determining its components, according to certain criteria which must be defined. Analyzing a function of one variable ($f(x)$) within a specific interval (Δx) using the SWM, one can determine the trains of square waves once the function in that interval is digitized; i.e., the components of the entity considered, according to the criterion established by the method.

In this article the generalization of the SWM was presented for the analysis of images.

In a given region when the SWM is applied to the analysis of an image, a digitized "version" of that image must first exist. Actually, the entity analyzed is the digitized image.

According to the notion of digitized image used here, that image could be considered to be a function of two variables, which is digitized, or made discrete, in a certain region.

The components that make up the digitized entity —the digitized image— are also, according to the criterion established by the SWM, trains of square waves. Each row and each column of the digitized image can be reconstructed by adding up all of the trains of square waves corresponding to that particular row or column. The entities analyzed here were actually sub-images of a given image. Each sub-image considered was a square which could be thought to be made up of 32 rows of 32 pixels each, or of 32 columns of 32 pixels each. Every row and every column of that digitized sub-image con be reconstructed by adding up 1,024 trains of square waves. Each one of these trains of square waves can be unambiguously determined by the SWM procedures specified. Reconstructing all the rows or all the columns of a digitized sub-image is equivalent to obtaining once again the value of the level of gray of each pixel in the given sub-image.

The SWM is a systematic method for analyzing any digitized image of the type considered, such as that of any digital photograph displayed using a gray scale.

Regarding the approach described, one limitation (pragmatically irrelevant but theoretically important) is the following: No proof was presented here of the existence and uniqueness of the solution of systems of linear algebraic equations such as (5).

From the computational viewpoint, applying the SWM to the analysis of images is simple and direct: Adequate software for the solution of algebraic linear equations already exists.

5.2 Perspectives

The generalization of the approach presented for digitized color images is immediate. The color corresponding to each pixel of the color image is achieved by the composition of three levels of color: those corresponding to three primary colors. Hence, any color image can be "decomposed" into three images: the first is done with one of the primary colors, the second with another of them and the third with the remaining primary color. Therefore, each of these three images may be analyzed using the method described here.

The level of gray for each of the pixels mentioned is achieved by composing the same levels of color for each of the three primary colors. Thus, for example, if one of the three primary colors is $(PC)_1$, another is $(PC)_2$, and the other $(PC)_3$, gray level 57 is obtained by composing level 57 of $(PC)_1$, $(PC)_2$, and $(PC)_3$. For this reason, for images made up of different levels of gray, it is not necessary to carry out three processes of analysis, one for each of the three color images mentioned. Those three analyses would lead to the same numerical results.

The use of the SWM for the analysis of color images will be discussed in further detail elsewhere.

Three other issues will also be addressed:

a. applying the SWM to the analysis of functions of n variables – $f(x_1, x_2, \ldots, x_n)$; this generalization would be useful for providing a geometric interpretation of the SWM applied to functions of any number ($n = 1, 2, \ldots, n$) of variables;

b. the role that the SWM can have in the numeric solution of differential equations of interest in mathematical physics (e.g.: Laplace's equation) with quite varied boundary conditions; and

c. certain applications of the SWM to the analysis of signals and images which are of interest in the field of biomedicine.

To the authors' best knowledge, the method presented here for image analysis has not been used previously. No references were found on the topic in the literature reviewed regarding image analysis and processing. In the vast amount of references, very diverse issues are addressed, such as image segmentation, image dilation, image erosion, image thresholding, image enhancement and image compression. A preliminary introduction to these topics may be found, for example, in [1], [6], [2] and [4]. As the SWM and its applications continue to be developed, it will be possible to determine whether it will be a significant contribution, complementary to the diverse methods existing for the analysis of signals and images.

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