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OPTIMAL CONTROL OF POLLUTION STOCK THROUGH ECOLOGICAL INTERACTION OF THE MANUFACTURER AND THE STATE

CONTROL ÓPTIMO DE CONTAMINACIÓN ALMACENADA A TRAVÉS DE INTERACCIÓN ECOLÓGICA ENTRE EL FABRICANTE Y EL ESTADO

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Abstract

A model of an interaction between a manufacturer and the state where the manufacturer produces a single product and the state controls the level of pollution is created and investigated. A local economy with a stock pollution problem that must choose between productive and environmental investments (control functions) is considered. The model is described by a nonlinear system of two differential equations with two bounded controls. The best optimal strategy is found analytically with the use of the Pontryagin Maximum Principle and Green's Theorem.

Keywords: optimal control, nonlinear model, environmental problem.

Resumen

Se ha creado e investigado un modelo de interaccin entre un fabricante y el estado donde el fabricante produce un solo producto y el estado controla el nivel de contaminación. Se considera una economía local con un problema de contaminación almacenada, que debe escoger entre inversiones en producción y medio ambiente (funciones de control). El modelo es descrito por un sistema de dos ecuaciones diferenciales con dos controles acotados. La mejor estrategia de control se encuentra analíticamente usando el Principio del Máximo de Pontryagin y el Teorema de Green.

Palabras clave: control óptimo, modelo no lineal, problema ambiental.

Mathematics Subject Classification: 49J15, 49N90, 93C10, 93C95.

1 Introduction

In the 21st century with the rapid growth of science and technology the man's economic activities began to produce an increasingly negative effect on the biosphere. Environmental pollution has become a significant obstacle to economic growth. The discharge of dust and gas into the atmosphere returns to the Earth in the form of acidic rain and affects crop, the quality of forests, the amount of fish. To this we can add the rise of chemicals, radioactivity, noise and other types of pollution. Economic, social, technological and biological processes have become so mutually dependent that modern production can usefully be considered as a complex economic system. Over the last several decades various concepts of environmental pollution control have been developed [1]-[3]. The most important of them can be summarized as follows.

1. International trade policy [4]-[7]. (Since production in one country will affect not only the environment within its borders, but also abroad, some countries cooperate on certain regulations and control of pollution and production.)

2. Local actions imposed by the state [8]-[10]. (Comparatively to the international trade policy, the most direct results are still in the hands of each individual country and local actions through the rules imposed by the state. In addition to pollution fines, some states impose taxes on the sales revenue and on total annual profit.)

3. Demands from the customers that prefer to buy from environmentally friendly companies [11].

Establishing optimal working conditions and control strategies is frequently accomplished with the aid of mathematical models [12, 13]. When a model is created, it is important to investigate it properly. There are very few analytical methods for solving optimal control problems for nonlinear models. In general, what unites all theses papers is that models are too simple (described by one equation with at most one unbounded control parameter).

We consider a local economy with a stock pollution problem that must choose between productive and environmental investments (control functions). It is similar to that in the work [9], but it is updated and more complex. For example, in our model two bounded controls depend on one another by an inequality constrain. This makes solving optimal control problem challenging.

Our paper deals with complete analysis of the considered optimal control problem. It is organized as follows. In Section 2, we discuss the model and the corresponding optimal control problem. For solving this problem we use two approaches: the Pontryagin Maximum Principle and Green's Theorem. Application of the Pontryagin Maximum Principle to the considered optimal control problem is discussed in Section 3. The types of the corresponding optimal controls are determined by the behavior of the switching functions. Therefore, analysis of the switching functions and discussion of the three cases arisen from this are considered in Section 4. Section 5 is devoted to obtaining results associated with Green's Theorem. Immediate finding of the types of controls, among which it is necessary to seek the optimal controls is given in Sections 6, 7. It is based on the facts, which have been established in Sections 4, 5. We proved that the optimal controls are piecewise constant functions with at most four switchings.

This allows us in Section 8 to reduce the considered optimal control problem to the problem of the finite dimensional optimization, for which the methods of the numerical solution are well developed. Section 9 presents our conclusions.

2 Description of the model

Consider a manufacturer producing a single product, which is always in demand. Its production activity causes of pollution. The state controls the levels of pollution by establishing the maximum allowance.

Let q be the cost of production funds, δ is the amortization coefficient and μ profitability of the manufacturer with initial production funds q_0. Then the volume of the production, given by the Cobb-Douglas production function at the level of production q can be found as $F(q) = \mu q_0^{(1-\alpha)} q^{\alpha}$, $\alpha \in (0, 1]$, where α is the elasticity coefficient. Assuming that it sells everything that it produces at the market price p, then the corresponding sales revenue can be found as $\Pi(q) = pF(q)$. Part of it, $\Pi(q)$, in the amount of $u_1\Pi(q) = pu_1F(q)$ will be invested into its own production and the other portion, $u_2\Pi(q) = pu_2F(q)$ will be invested into cleaning the environment. The remaining amount will be kept by the manufacturer in the amount of $(1 - u_1 - u_2)\Pi(q) = p(1 - u_1 - u_2)F(q)$. Here u_1 and u_2 give the actual portion of each investment.

Let s be the pollution stock from the production activity of the manufacturer. On one hand, this pollution stock will increase proportionally to the volume of production $rF(q)$. On the other hand, it will decrease with the rate of natural pollution degradation σ and additionally as a result of the investment of the manufacturer into the environmental cleaning with rate $\beta pu_2F(q)$. Here r, σ, β are coefficients of proportionality. We will consider the situation when the effectiveness of the pollution control depends on the funds invested into environmental cleaning.

Therefore, the dynamics of production funds $q(t)$ and pollution stock $s(t)$ on the given time interval $[0, T]$ will be described by the following nonlinear system of two differential equations

$$\begin{cases} \dot{q}(t) = -\delta q(t) + pu_1(t)F(q(t)),\ t \in [0, T], \\ \dot{s}(t) = -\sigma s(t) + (r - \beta pu_2(t))F(q(t)), \\ q(0) = q_0,\ s(0) = s_0;\ q_0, s_0 > 0. \end{cases} \tag{2.1}$$

The right side of these equations contains control functions $u_1(t)$, $u_2(t)$

satisfying the following restrictions:

$$u_1(t) \geq 0,\ u_2(t) \geq 0,\ u_1(t) + u_2(t) \leq 1,\ t \in [0, T]. \tag{2.2}$$

Let p_0 be the unit price of the produced good. The manufacturer's profit can be written as

$$p(1 - u_1 - u_2)F(q) - p_0 F(q) - \lambda g(s). \tag{2.3}$$

In this expression the value $p_0 F(q)$ is the production expense, the value $\lambda g(s)$ is the penalty (fine) paid by the manufacturer if it exceeds the maximum level of pollution, c. The value of c is determined by the state and it is assumed that $g(s) = 0.5 \cdot (\max\{0, s - c\})^2$ and λ is the coefficient of proportionality.

Using (2.3) we can define the manufacturer's objective function as

$$I(u_1, u_2) = \gamma e^{-\rho T} q(T) + \tag{2.4}$$

$$+ \int_0^T e^{-\rho t} \Big\{ p(1 - u_1(t) - u_2(t))F(q(t)) - p_0 F(q(t)) - \lambda g(s(t)) \Big\} dt,$$

where ρ is the discount coefficient, and $\gamma \in (0, 1)$ is a weighting coefficient. The relationship (2.4) is the discounted wealth of the manufacturer (weighted sum of discounted cost of the production funds at the final time T and the discounted cumulative profit at the time interval $[0, T]$).

Then, we define the control set $D(T)$ as the set of all pairs of Lebesgue measurable functions, $(u_1(t), u_2(t))$, satisfying the inequalities (2.2).

Therefore, for the system of differential equations (2.1) we will consider the problem of maximizing the objective function (2.4) on the set of all admissible controls $D(T)$.

Given specifications of the formulated optimal control problem we will study only the situation when $\delta = 0$, $\sigma = 0$, $\rho = 0$ and $\alpha = 1$, i.e. we investigate the case of the linear production function, $F(q) = \mu q$. We ignore the amortization of the production funds and neglect the natural degradation of the pollution.

Let l be the positive value, such that the equality $r = \beta p l$ is valid. Then, the second equation of the system (2.1) can be rewritten as

$$\dot{s}(t) = \mu \beta p (l - u_2(t)) q(t).$$

In this equation the value l reflects the ability of the invested funds to change the pollution stock s. If $l > 1$, then the invested funds never reduce the pollution stock s. If $l = 1$, then the pollution stock s may retain at the same reached level. In fact, only if $l < 1$ the invested funds can reduce the pollution stock s.

Additionally, we suppose that the inequality

$$(1 - l) \neq \frac{p_0}{p}$$

is valid.

Putting the terminal term of the objective function (2.4) under the sign of the integral, we have the following optimal control problem:

$$\begin{cases} \dot{q}(t) = \mu p u_1(t) q(t), \ t \in [0, T], \\ \dot{s}(t) = \mu \beta p (l - u_2(t)) q(t), \\ q(0) = q_0, \ s(0) = s_0; \ q_0, s_0 > 0, \end{cases} \tag{2.5}$$

$$J(u_1, u_2) = \int_0^T \Big\{ \mu \Big(p(1 - (1-\gamma)u_1(t) - u_2(t)) - p_0 \Big) q(t) - \tag{2.6}$$

$$- \lambda g(s(t)) \Big\} dt \to \max_{(u_1(\cdot), u_2(\cdot)) \in D(T)}.$$

The statement below describes the properties of the variables $q(t)$, $s(t)$ for the system (2.5), which can be easily proven using direct integration of the equations of this system.

Lemma 1 *Let $(u_1(\cdot), u_2(\cdot)) \in D(T)$ be some control functions. Then the solutions $q(t)$, $s(t)$ of the system (2.5) for all $t \in [0, T]$ satisfy the inequalities:*

$$q(t) > 0 \quad \text{for} \quad \forall\, l,$$
$$s(t) > 0 \quad \text{for} \quad l \geq 1.$$

From the analysis of the equations of the system (2.5) we have the following statement.

Lemma 2 *Let $(u_1(\cdot), u_2(\cdot)) \in D(T)$ be some control functions. Then for the derivatives of the functions $q(t)$, $s(t)$ for almost all $t \in [0, T]$ the following inequalities hold:*

$$\dot{q}(t) \geq 0 \quad \text{for} \quad \forall\, l,$$
$$\dot{s}(t) > 0 \quad \text{for} \quad l > 1,$$
$$\dot{s}(t) \geq 0 \quad \text{for} \quad l = 1.$$

The existence of the optimal controls $(u_1^*(t), u_2^*(t))$ and the corresponding optimal trajectory $(q_*(t), s_*(t))$ for the problem (2.5),(2.6) follows from Filippov Theorem [14].

3 Pontryagin Maximum Principle

In order to solve the problem (2.5),(2.6) we will apply Pontryagin Maximum Principle [15]. For the optimal controls $(u_1^*(t), u_2^*(t))$ and the corresponding trajectory $(q_*(t), s_*(t))$ there exist the nontrivial solutions $\psi_*(t)$, $\phi_*(t)$ of the adjoint system

$$\begin{cases} \dot{\psi}_*(t) = -\mu p u_1^*(t)\psi_*(t) - \mu\beta p(l - u_2^*(t))\phi_*(t) - \\ \qquad - \mu(p(1-(1-\gamma)u_1^*(t) - u_2^*(t)) - p_0),\ t \in [0,T], \\ \dot{\phi}_*(t) = \lambda \dot{g}(s_*(t)), \\ \psi_*(T) = 0,\ \phi_*(T) = 0, \end{cases} \tag{3.1}$$

for which the following relationship is valid

$$(u_1^*(t), u_2^*(t)) = \begin{cases} (0,0), & \text{if } L_{u_1}(t) < 0,\ L_{u_2}(t) < 0, \\ (0,1), & \text{if } L_{u_2}(t) > 0,\ L_{u_2}(t) > L_{u_1}(t), \\ u_1 = 0,\ \forall u_2 \in [0,1], & \text{if } L_{u_2}(t) = 0,\ L_{u_1}(t) < 0, \\ (1,0), & \text{if } L_{u_1}(t) > 0,\ L_{u_2}(t) < L_{u_1}(t), \\ \forall u_1 \in [0,1],\ u_2 = 0, & \text{if } L_{u_1}(t) = 0,\ L_{u_2}(t) < 0, \\ \forall (u_1, u_2): & \\ u_1 \ge 0,\ u_2 \ge 0, & \\ u_1 + u_2 = 1, & \text{if } L_{u_1}(t) > 0,\ L_{u_1}(t) = L_{u_2}(t), \\ \forall (u_1, u_2): & \\ u_1 \ge 0,\ u_2 \ge 0, & \\ u_1 + u_2 \le 1, & \text{if } L_{u_1}(t) = 0,\ L_{u_2}(t) = 0, \end{cases} \tag{3.2}$$

where

$$L_{u_1}(t) = \psi_*(t) - (1-\gamma),\ L_{u_2}(t) = -\beta\phi_*(t) - 1,\ t \in [0,T]. \tag{3.3}$$

Functions $L_{u_1}(t)$, $L_{u_2}(t)$ are called switching functions. We can see from (3.2) that their behavior determines the type of the optimal controls $(u_1^*(t), u_2^*(t))$.

Moreover, the Hamilton-Pontryagin function

$$H(q_*(t), s_*(t), \psi_*(t), \phi_*(t), u_1^*(t), u_2^*(t))$$

corresponding to the systems (2.5),(3.1) is constant on the time interval $[0,T]$, i.e. the following relationship holds

$$\mu p u_1^*(t) q_*(t)\psi_*(t) + \mu\beta p(l - u_2^*(t)) q_*(t)\phi_*(t) + \tag{3.4}$$

$$+\mu(p(1-(1-\gamma)u_1^*(t)-u_2^*(t))-p_0)q_*(t)-\lambda g(s_*(t))=Q_*,\ t\in[0,T].$$

The systems of equations (2.5),(3.1) and the relationships (3.2),(3.3) form a two-point boundary value problem for the Maximum Principle. We will study this problem in depth.

First, let us obtain phase portraits for system of equations (2.5) with controls $(u_1^*(t),u_2^*(t))$ from the relationship (3.2). There are three possible situations:

a) Let $(u_1^*(t),u_2^*(t))=(0,0)$. The system of equations (2.5) will be written as

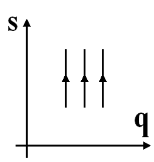
$$\dot{q}_*(t)=0,\ \dot{s}_*(t)=\mu\beta plq_*(t).$$

The phase portrait of the system is shown in Figure 1.

b) Let $(u_1^*(t),u_2^*(t))=(1,0)$. The system of equations (2.5) will be written as

$$\dot{q}_*(t)=\mu pq_*(t),\ \dot{s}_*(t)=\mu\beta plq_*(t).$$

From these relationships we obtain the equality $\frac{ds_*}{dq_*}=\beta l>0$. The phase portrait is shown in Figure 2.



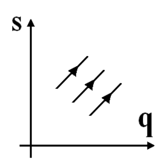


Figure 1: (Left) Phase portrait for $(u_1^*(t),u_2^*(t))=(0,0)$.

Figure 2: (Right) Phase portrait for $(u_1^*(t),u_2^*(t))=(1,0)$.

c) Let $(u_1^*(t),u_2^*(t))=(0,1)$. The system of equations (2.5) will be written as

$$\dot{q}_*(t)=0,\ \dot{s}_*(t)=\mu\beta p(l-1)q_*(t).$$

The phase portrait of the system depending on the value l is shown in Figure 3.

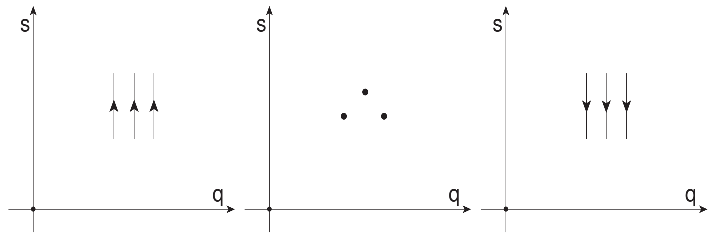


Figure 3: Phase portrait for $(u_1^*(t), u_2^*(t)) = (0,1)$ for $l > 1$, $l = 1$, and $l < 1$.

Next, from the systems of equations (3.1) and formulas (3.3) we will obtain for the switching functions $L_{u_1}(t)$, $L_{u_2}(t)$ the following Cauchy problems:

$$\begin{cases} \dot{L}_{u_1}(t) = -\mu p u_1^*(t) L_{u_1}(t) + \mu p(l - u_2^*(t)) L_{u_2}(t) - \mu((1-l)p - p_0), \\ L_{u_1}(T) = -(1-\gamma), \end{cases} \tag{3.5}$$

$$\begin{cases} \dot{L}_{u_2}(t) = -\lambda \beta \dot{g}(s_*(t)), \\ L_{u_2}(T) = -1. \end{cases} \tag{3.6}$$

From the analysis of the Cauchy problems (3.5),(3.6) we see that the following statements are true.

Lemma 3 *The switching function $L_{u_2}(t)$ is a non-increasing function on the interval $[0, T]$.*

Lemma 4 *For the switching functions $L_{u_1}(t)$, $L_{u_2}(t)$ there exists such moment of time $\tau \in [0, T)$, that simultaneously the following inequalities are valid:*

$$L_{u_1}(t) < 0,\ L_{u_2}(t) < 0,\ t \in (\tau, T]. \tag{3.7}$$

Moreover, the interval $(\tau, T]$ is of maximum length and cannot be reduced. Otherwise, at least one of the inequalities (3.7) will not hold.

From Lemma 4 and relationship (3.2) it follows that the formula below for the optimal controls $(u_1^*(t), u_2^*(t))$ will be valid

$$(u_1^*(t), u_2^*(t)) = (0, 0),\ t \in (\tau, T]. \tag{3.8}$$

We will conduct further analysis of the controls $(u_1^*(t), u_2^*(t))$ in the reversed time interval, i.e. from time $t = T$ to time $t = 0$. For this, we will consider a point (q_T, s_T) which is the end of the optimal trajectory $(q_*(t), s_*(t))$. Thus, the equalities are valid:

$$q_T = q_*(T),\ s_T = s_*(T).$$

Next, the statement is true.

Lemma 5 *For the switching function $L_{u_1}(t)$ one of the following cases holds:*

1) Let $Q_* > 0$, then for all $t \in [0,T]$ the inequality $\dot{L}_{u_1}(t) < 0$ is valid. It means that the switching function $L_{u_1}(t)$ is a decreasing function;

2) Let $Q_* = 0$, then the relationship

$$\dot{L}_{u_1}(t) \begin{cases} = 0, & \text{if} \quad s_*(t) \leq c, \\ < 0, & \text{if} \quad s_*(t) > c, \end{cases}$$

is valid. It means that the switching function $L_{u_1}(t)$ is a non-increasing function.

3) Let $Q_* < 0$, then, first, there exists the moment of time $\tau \in [0,T)$ that on the interval $(\tau, T]$ the switching function $L_{u_1}(t)$ decreases. Second, the inequality $\dot{L}_{u_1}(t) > 0$ is valid under the condition $s_*(t) \leq c$, which means that the function $L_{u_1}(t)$ is an increasing function.

Proof. From formulas (3.3),(3.4) and Cauchy problem (3.5) we find the equality

$$Q_* + \lambda g(s_*(t)) = -q_*(t)\dot{L}_{u_1}(t),\ t \in [0,T].$$

From analysis of this expression, Lemmas 1, 4 and properties of the function $g(s)$ the validity of this statement follows. ■

Now, we will find the conditions, for which the value Q_* is positive, negative or equal to zero. For this, at formula (3.4) we let $t = T$, use Lemma 4 and initial conditions from the Cauchy problems (3.5),(3.6). We obtain the equality

$$Q_* = \mu(p - p_0)q_T - \lambda g(s_T).$$

From this expression we find that the inequality $Q_* > 0$ holds under the condition

$$q_T > \frac{\lambda}{\mu(p - p_0)} g(s_T).$$

Corresponding region is shown in Figure 4 as horizontal hatched.

The inequality $Q_* < 0$ is valid under the condition

$$q_T < \frac{\lambda}{\mu(p - p_0)} g(s_T).$$

Corresponding region is shown in Figure 4 as vertical hatched.

At last, the equality $Q_* = 0$ holds under the condition

$$q_T = \frac{\lambda}{\mu(p - p_0)} g(s_T).$$

Corresponding region is shown in Figure 4 as bold line.

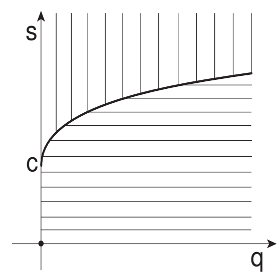


Figure 4: Regions for $Q_* > 0$, $Q_* < 0$, and $Q_* = 0$.

Next, the statement is true.

Lemma 6 *There is no interval $\Delta \subset [0, T]$, on which simultaneously the following equalities are valid:*

$$L_{u_1}(t) = 0,\ L_{u_2}(t) = 0. \tag{3.9}$$

Proof. We prove this statement by contradiction. Let us assume, that there exists the interval $\Delta \subset [0, T]$, on which simultaneously the equalities (3.9) hold. Then from the Cauchy problem (3.5) it follows the contradictory equality $(1 - l)p = p_0$. The proof is complete. ■

Now, we will consider the situation when $s_T \leq c$. We have the statement.

Lemma 7 *Let $s_T \leq c$, then the optimal controls $(u_1^*(t), u_2^*(t))$ have one of the following types:*

$$(u_1^*(t), u_2^*(t)) = (0, 0),\ t \in [0, T], \tag{3.10}$$

$$(u_1^*(t), u_2^*(t)) = \begin{cases} (1,0), & \text{if} \quad 0 \le t \le \theta, \\ (0,0), & \text{if} \quad \theta < t \le T, \end{cases} \tag{3.11}$$

where $\theta \in (0, T)$ *is the moment of switching.*

Proof. At first, we show that the restriction $s_*(t) \le c$ is valid for all $t \in [0, T]$. If $l \ge 1$ the validity of the last inequality follows from Lemma 2.

Now, we will consider the case $l < 1$. We assume the contradiction. Let there exist the moment of time $\theta \in (0, T)$, for which at $t < \theta$ the inequality $s_*(t) > c$ holds and for $t \in [\theta, T]$ the inequality $s_*(t) \le c$ is valid. Then the small value $\epsilon > 0$ is found that on the interval $(\theta - \epsilon, \theta + \epsilon)$ from the relationship (3.2) the optimal controls $(u_1^*(t), u_2^*(t))$ take values $(0, 1)$. It means that the inequality below holds

$$L_{u_2}(t) > 0, \ t \in (\theta - \epsilon, \theta + \epsilon).$$

Hence, from the continuity of the switching function $L_{u_2}(t)$ we have the inequality

$$L_{u_2}(\theta + \epsilon) \ge 0. \tag{3.12}$$

From the other side, from our assumption the following equality holds

$$\dot{g}(s_*(t)) = 0, \ t \in [\theta, T].$$

From this expression and the Cauchy problem (3.6) we have the equality

$$L_{u_2}(t) = -1, \ t \in [\theta, T].$$

Then we find the expression $L_{u_2}(\theta + \epsilon) = -1$, which contradicts the inequality (3.12). Our assumption was wrong. The inequality below holds

$$s_*(t) \le c, \ t \in [0, T].$$

From this relationship and the Cauchy problem (3.6) we have the equality

$$L_{u_2}(t) = -1, \ t \in [0, T]. \tag{3.13}$$

At $s_T \le c$ the inequality $Q_* > 0$ is valid. Then from Lemma 5 we conclude the decrease of the switching function $L_{u_1}(t)$. The consequent behavior of the function $L_{u_1}(t)$ depends on the value $L_{u_1}(0)$.

If $L_{u_1}(0) \le 0$, then for all $t \in (0, T]$ we have the inequality

$$L_{u_1}(t) < 0. \tag{3.14}$$

From the relationship (3.2) and expressions (3.13),(3.14) the formula (3.10) follows.

If $L_{u_1}(0) > 0$, then there exists the moment of time $\theta \in (0, T)$ that the following relationship holds

$$L_{u_1}(t) \begin{cases} > 0, & \text{if} \quad 0 \leq t < \theta, \\ = 0, & \text{if} \quad t = 0, \\ < 0, & \text{if} \quad \theta < t \leq T. \end{cases} \tag{3.15}$$

From the relationship (3.2) and expressions (3.13),(3.15) we have the formula (3.11). The statement is proved. ■

Further, we will consider the situation when $s_T > c$.

4 Analysis of the switching functions

Let us consider the relationship (3.2). From it's analysis we see that the phase plane of the switching functions (L_{u_1}, L_{u_2}) is divided into three regions (see Figure 5). In the region $L_{u_1} < 0$, $L_{u_2} < 0$ the optimal controls $(u_1^*(t), u_2^*(t))$ are defined uniquely by values $(0, 0)$. In the region $L_{u_2} > 0$, $L_{u_2} > L_{u_1}$ these controls also are defined uniquely by values $(0, 1)$. At last, in the region $L_{u_1} > 0$, $L_{u_2} < L_{u_1}$ the optimal controls $(u_1^*(t), u_2^*(t))$ are defined uniquely by the values $(1, 0)$.

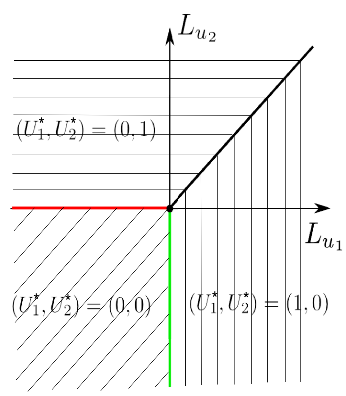


Figure 5: Phase plane of switching functions.

However, on the line $L_{u_2} = 0$, $L_{u_1} < 0$ the value $u_1^*(t) = 0$, and the value $u_2^*(t)$ is not uniquely defined. Similarly, on the line $L_{u_1} = 0$,

$L_{u_2} < 0$ the value $u_2^*(t) = 0$ and the value $u_1^*(t)$ is not unique. At last, on the line $L_{u_1} = L_{u_2} > 0$ the equality $u_1^*(t) + u_2^*(t) = 1$ and the values $(u_1^*(t), u_2^*(t))$ are not uniquely defined. At the origin, $L_{u_1} = L_{u_2} = 0$, the Pontryagin Maximum Principle does not give information about the optimal controls $(u_1^*(t), u_2^*(t))$. In all these situations at controls $(u_1^*(t), u_2^*(t))$ singular regimes [16] are possible.

At once, we note that from Lemma 6 it follows the impossibility of the singular regime at the last situation.

The optimal controls $(u_1^*(t), u_2^*(t))$ and corresponding trajectory $(q_*(t), s_*(t))$ in the phase plane (L_{u_1}, L_{u_2}) produce a curve of switching functions $(L_{u_1}(t), L_{u_2}(t))$, $t \in [0, T]$. From the analysis of the values $L_{u_1}(T)$, $L_{u_2}(T)$ we know that this curve ends in the third quadrant at the point with coordinates $(-(1-\gamma), -1)$.

It follows from Lemma 4 that the time moment τ is such that the curve of switching functions $(L_{u_1}(t), L_{u_2}(t))$ intersects either the line $L_{u_2} = 0$, $L_{u_1} < 0$, or the line $L_{u_1} = 0$, $L_{u_2} < 0$, or goes to the origin $L_{u_1} = L_{u_2} = 0$.

Therefore, these three cases will be considered.

Case 1. Let on some interval $\Delta \subset [0, T]$ the curve of switching functions $(L_{u_1}(t), L_{u_2}(t))$ corresponding to the optimal controls $(u_1^*(t), u_2^*(t))$ and optimal trajectory $(q_*(t), s_*(t))$ be located in the region where $u_2^*(t) = 0$. This region is shown in Figure 6 as hatched.

We note that on the line $L_{u_1} = 0$, $L_{u_2} < 0$, where the Pontryagin Maximum Principle is not unique, the optimal trajectory $(q_*(t), s_*(t))$ with the controls $(u_1^*(t), u_2^*(t))$ may have a singular regime in the hatched region. Moreover, one of these intervals Δ is the interval (τ, T) from Lemma 4.

Case 2. Let on some interval $\Delta \subset [0, T]$ the curve of switching functions $(L_{u_1}(t), L_{u_2}(t))$ corresponding to the optimal controls $(u_1^*(t), u_2^*(t))$ and optimal trajectory $(q_*(t), s_*(t))$ be located in the region where $u_1^*(t) = 0$. This region is shown in Figure 7 as hatched.

We note that on the line $L_{u_2} = 0$, $L_{u_1} < 0$, where the Pontryagin Maximum Principle is not unique, the optimal trajectory $(q_*(t), s_*(t))$ with the controls $(u_1^*(t), u_2^*(t))$ may have a singular regime in the hatched region. Moreover, one of these intervals Δ is the interval (τ, T) from Lemma 4.

Case 3. Let on some interval $\Delta \subset [0, T]$ the curve of switching functions $(L_{u_1}(t), L_{u_2}(t))$ corresponding to the optimal controls $(u_1^*(t), u_2^*(t))$ and optimal trajectory $(q_*(t), s_*(t))$ is located into the region, where these controls satisfy the equality $u_1^*(t) + u_2^*(t) = 1$. This region is shown in Fig-

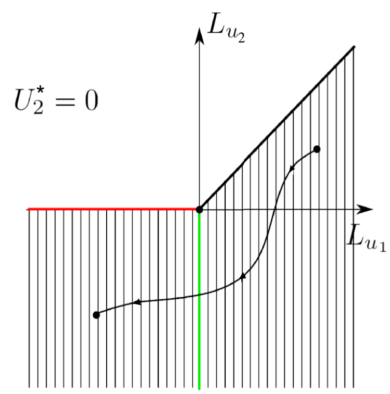


Figure 6: Region for $u_2^*(t) = 0$ (Case 1).

ure 8 as hatched.

We note that on the line $L_{u_1} = L_{u_2} > 0$, where the Pontryagin Maximum Principle is not unique, the optimal trajectory $(q_*(t), s_*(t))$ with the controls $(u_1^*(t), u_2^*(t))$ may have a singular regime in the hatched region.

5 Green's Theorem

While investigating the Cases 1-3 we will use the study of Cauchy problems (3.4),(3.5) and the local analysis of the optimal trajectories developed in [12, 17].The foundation of this analysis is the use of Green's Theorem.

In the Case 1 we have the following arguments. On the interval $\Delta \subset [0,T]$ under the condition $u_2^*(t) = 0$ the controls $u_1^*(t)$, $1 - u_1^*(t)$ can be expressed from the system (2.5) as follows:

$$u_1^*(t) = \frac{\dot{q}_*(t)}{\mu p q_*(t)},\ 1 - u_1^*(t) = \frac{\dot{s}_*(t) - \beta l \dot{q}_*(t)}{\mu \beta p l q_*(t)}.$$

Next, we substitute these expressions into the functional (2.6). After necessary transformations we obtain the relationship

$$J(u_1^*, u_2^*) = \int_\Delta \left(-(1-\gamma)\dot{q}_*(t) + \frac{\mu(p-p_0)q_*(t) - \lambda g(s_*(t))}{\mu \beta p l q_*(t)} \dot{s}_*(t) \right) dt + \dots.$$

We have the contour integral at the optimal trajectory $(q_*(t), s_*(t))$ of the type

$$J(u_1^*, u_2^*) = \int_{C_*} -(1-\gamma)dq_* + \frac{\mu(p-p_0)q_* - \lambda g(s_*)}{\mu \beta p l q_*} ds_* + \dots,$$

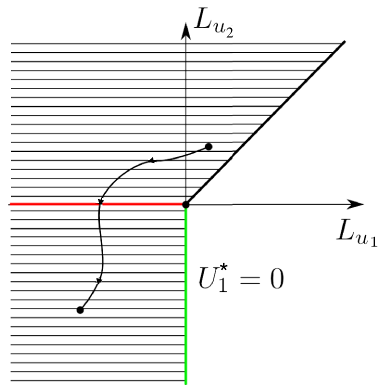


Figure 7: Region for $u_1^*(t) = 0$ (Case 2).

where C_* is the curve determined by this trajectory.

Now, at the optimal trajectory $(q_*(t), s_*(t))$ on the small interval $\omega \subset \Delta$ we will replace the control $u_1^*(t)$ as shown in Figure 9. We consider the difference of corresponding values of the functionals. One of which corresponds to the optimal trajectory $(q_*(t), s_*(t))$ that contains the path ABC, and the other to the same trajectory $(q_*(t), s_*(t))$ that contains the path ADC. We have the chain of equalities:

$$J_{ABC}(u_1^*, u_2^*) - J_{ADC}(u_1^*, u_2^*) =$$

$$= \oint_{ABCDA} -(1-\gamma)dq_* + \frac{\mu(p-p_0)q_* - \lambda g(s_*)}{\mu\beta plq_*} ds_* = \iint_{\Omega} \frac{\lambda g(s_*)}{\mu\beta plq_*^2} dq_* ds_* \geq 0.$$

Here Green's Theorem was used and Ω is the region bounded by the closed path $ABCDA$. The shape of this closed path is based on the phase portraits of the system (2.5) shown in Figures 1, 2. From the analysis of the last expression, we see that in the situation when the procedure of substituting of the control $u_1^*(t)$ on the interval $\omega \subset \Delta$ is made in the region $s < c$, the values of the corresponding functionals are equal. So, we cannot say anything about the considered difference. In the situation, when the procedure of substituting of the control $u_1^*(t)$ is made in the region $s > c$, the difference of the values of the functionals is positive.

Based on these, we can support the following statement.

Lemma 8 *Let $(q_*(t), s_*(t))$ be the optimal trajectory, for which on some interval $\Delta \subset [0, T]$ the inequality $s_*(t) > c$ holds and the corresponding*

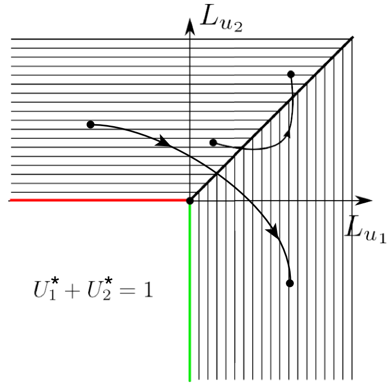


Figure 8: Region for $u_1^*(t) + u_2^*(t) = 1$ (Case 3).

optimal controls $(u_1^*(t), u_2^*(t))$ *obey the equality* $u_2^*(t) = 0$. *Therefore, the control* $u_1^*(t)$ *on the interval* Δ *is one of the following types:*

either takes the constant value from $\{0; 1\}$,

or it has the moment of switching $\theta \in \Delta$, *for which the following equalities are valid:*

$$u_1^*(\theta - 0) = 1,\ u_1^*(\theta + 0) = 0.$$

Proof. We assume the contradiction. Let the following equalities hold:

$$u_1^*(\theta - 0) = 0,\ u_1^*(\theta + 0) = 1.$$

Using arguments presented above for the optimal value of functional $J_* = J(u_1^*, u_2^*)$ we find the chain of equalities:

$$J_* - J_{ABC}(u_1^*, u_2^*) = J_{ADC}(u_1^*, u_2^*) - J_{ABC}(u_1^*, u_2^*) =$$

$$= -(J_{ABC}(u_1^*, u_2^*) - J_{ADC}(u_1^*, u_2^*)) < 0,$$

which is contradictory. Our assumption was wrong. The statement is proved. ■

Remark 1 *Lemma 8 is both valid for all points on the line* $s = c$ *and for the points below this line, in its small neighborhood.*

Remark 2 *The statement of Lemma 8 does not depend on the value* l.

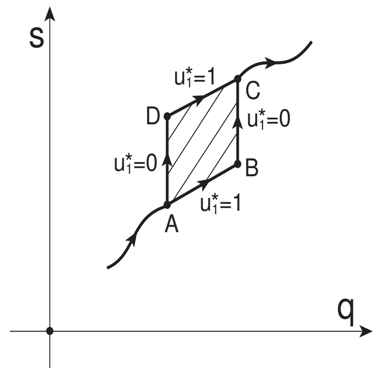


Figure 9: Optimal trajectory $(q_*(t), s_*(t))$ in Case 1 with controls $u_1^*(t) =\ 0$, $u_1^*(t) = 1$ on small interval ω.

In the Case 2 we have the following arguments. From the first equation of the system (2.5) it follows that the function $q_*(t)$ does not change its value and $q_*(t) = q_*$, $t \in \Delta$. Therefore, on the interval $\Delta \subset [0, T]$ under the condition $u_1^*(t) = 0$ the control $u_2^*(t)$ can be expressed from the system (2.5) as follows

$$u_2^*(t) = l - \frac{\dot{s}_*(t)}{\mu\beta p q_*}.$$

Next, we substitute this expression into the functional (2.6). After necessary transformations we obtain the relationship

$$J(u_1^*, u_2^*) = \int_\Delta \left(\mu((1-l)p - p_0)q_* - \lambda g(s_*(t)) + \frac{1}{\beta}\dot{s}_*(t) \right) dt + \dots\ .$$

We have the contour integral at the optimal trajectory $(q_*, s_*(t))$ of the type

$$J(u_1^*, u_2^*) = \int_{C_*} (\mu((1-l)p - p_0)q_* - \lambda g(s_*))dt + \frac{1}{\beta} ds_* + \dots,$$

where C_* is the curve determined by this trajectory.

Now, at the trajectory $(t, s_*(t))$ on the small interval $\omega \subset \Delta$ we will replace the control $u_2^*(t)$ as shown in Figure 10. We consider the difference of corresponding values of the functionals. One of which corresponds to the trajectory $(t, s_*(t))$ that contains the path ABC, and the other to the same trajectory $(t, s_*(t))$ that contains the path ADC. We have the chain of equalities:

$$J_{ABC}(u_1^*, u_2^*) - J_{ADC}(u_1^*, u_2^*) =$$

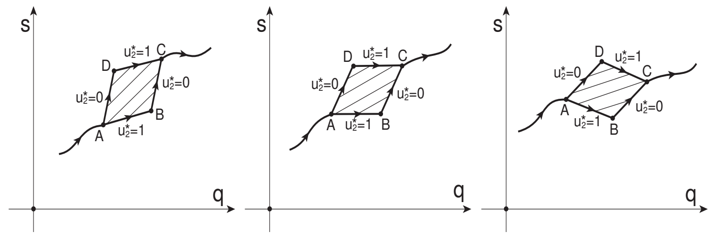


Figure 10: Optimal trajectory $(q_*(t), s_*(t))$ in Case 2 with controls $u_2^*(t) = 0$, $u_2^*(t) = 1$ on small interval ω for $l > 1$, $l = 1$, and $l < 1$.

$$= \oint_{ABCDA} (\mu((1-l)p - p_0)q_* - \lambda g(s_*))dt + \frac{1}{\beta} ds_* = \iint_{\Omega} \lambda \dot{g}(s_*) dt ds_* \geq 0.$$

Here Green's Theorem was used and Ω is the region bounded by the closed path $ABCDA$. The shape of this closed path is based on the phase portraits of the system (2.5) shown in Figures 1, 3. From the analysis of the last expression, we see that in the situation when the procedure of substituting of the control $u_2^*(t)$ on the interval $\omega \subset \Delta$ is made in the region $s < c$, the values of the corresponding functionals are equal. So, we cannot say anything about the considered difference. In the situation, when the procedure of substituting of the control $u_2^*(t)$ is made in the region $s > c$, the difference of the values of the functionals is positive.

Based on these, we can support the following statement.

Lemma 9 *Let $(q_*(t), s_*(t))$ be the optimal trajectory, for which on some interval $\Delta \subset [0, T]$ the inequality $s_*(t) > c$ holds and the corresponding optimal controls $(u_1^*(t), u_2^*(t))$ obey the equality $u_1^*(t) = 0$. Then the control $u_2^*(t)$ on the interval Δ is one of the following types:*

either takes the constant value from $\{0; 1\}$,

or it has the moment of switching $\theta \in \Delta$, for which the following equalities are valid:

$$u_2^*(\theta - 0) = 1, \; u_2^*(\theta + 0) = 0.$$

Proof. We assume the contradiction. Let the following equalities hold:

$$u_2^*(\theta - 0) = 0, \; u_2^*(\theta + 0) = 1.$$

Using arguments presented above for the optimal value of functional $J_* = J(u_1^*, u_2^*)$ we find the chain of equalities:

$$J_* - J_{ABC}(u_1^*, u_2^*) = J_{ADC}(u_1^*, u_2^*) - J_{ABC}(u_1^*, u_2^*) =$$

$$= -(J_{ABC}(u_1^*, u_2^*) - J_{ADC}(u_1^*, u_2^*)) < 0,$$

which is contradictory. Our assumption was wrong. The statement is proved. ■

Remark 3 *Lemma 9 is both valid for all points on the line $s = c$ and for the points below this line, in its small neighborhood.*

Remark 4 *The statement of Lemma 9 does not depend on the value l.*

6 Behavior of the optimal solutions at $l \geq 1$

Now, we will consider the situation $l \geq 1$. For the Case 3 we will study the behavior of the curve of the switching functions $(L_{u_1}(t), L_{u_2}(t))$ with respect to the line $L_{u_1} = L_{u_2} > 0$. For this, in the region $L_{u_1} > 0$, $L_{u_2} > 0$ we introduce the auxiliary function $L_0(t) = L_{u_1}(t) - L_{u_2}(t)$, and using the Cauchy problems (3.5),(3.6) we write for this function the corresponding differential equation

$$\dot{L}_0(t) = -\mu p(l - u_2^*(t))L_0(t) + \Big\{\mu p(l-1)L_{u_1}(t) + \lambda\beta\dot{g}(s_*(t)) - \mu((1-l)p - p_0)\Big\}. \tag{6.1}$$

At the considered region the expression inside the braces is positive. Therefore, from the analysis of the equation (6.1) we have the statement.

Lemma 10 *In the region $L_{u_1} > 0$, $L_{u_2} > 0$ the auxiliary function $L_0(t)$ has at most one zero. Moreover, if for some value $\theta \in (0, T)$ the equality $L_0(\theta) = 0$ holds, then the inequality $\dot{L}_0(\theta) > 0$ is valid.*

From Lemma 10 it follows that at the line $L_{u_1} = L_{u_2} > 0$ a singular regime is impossible. The curve of the switching functions $(L_{u_1}(t), L_{u_2}(t))$ stays either in the region $L_{u_1} < L_{u_2}$, or in the region $L_{u_1} > L_{u_2}$. If this curve intersects the line $L_{u_1} = L_{u_2} > 0$, then definitely from the region $L_{u_1} < L_{u_2}$ to the region $L_{u_1} > L_{u_2}$ (see Figure 11).

In the same way, we have the statement, which describes the behavior of the switching function $L_{u_1}(t)$. Its validity follows from the analysis of the Cauchy problem (3.5).

Lemma 11 *If on the interval $\Delta \subset [0, T]$ the inequality $L_{u_2}(t) \geq 0$ is valid, then the switching function $L_{u_1}(t)$ has at most one zero. Moreover, if for some value $\theta \in \Delta$ the equality $L_{u_1}(\theta) = 0$ holds, then the inequality $\dot{L}_{u_1}(\theta) > 0$ is valid.*

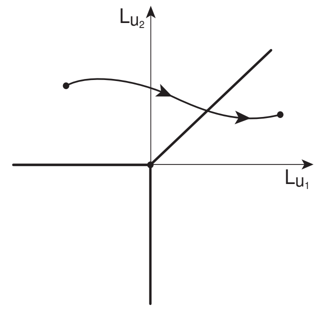


Figure 11: Behavior of the curve of switching functions $(L_{u_1}(t), L_{u_2}(t))$ when it intersects the line $L_{u_1} = L_{u_2} > 0$.

Analogous arguments hold for the situation, when at the moment of time $\tau \in (0, T)$ the curve of the switching functions $(L_{u_1}(t), L_{u_2}(t))$ comes to the origin, i.e. the following equalities are valid:

$$L_{u_1}(\tau) = 0, \; L_{u_2}(\tau) = 0.$$

Indeed, the differential equation similar to the equation (6.1) has the type

$$\dot{L}_0(t) = -\mu p(l - u_2^*(t))L_0(t) +$$

$$\Big\{\mu p(l - u_1^*(t) - u_2^*(t))L_{u_1}(t) + \lambda\beta\dot{g}(s_*(t)) - \mu((1-l)p - p_0)\Big\}.$$

From this equation we find the inequality $\dot{L}_0(\tau) > 0$. Then at the moment of time τ the curve of the switching functions $(L_{u_1}(t), L_{u_2}(t))$ moves from the region $L_{u_1} < L_{u_2}$ to the region $L_{u_1} > L_{u_2}$ (see Figure 12).

Since, outside the third quadrant the curve of the switching functions $(L_{u_1}(t), L_{u_2}(t))$ stays above the line $L_{u_1} = L_{u_2}$, then from the relationship (3.2) for the optimal controls $(u_1^*(t), u_2^*(t))$ we obtain the type

$$(u_1^*(t), u_2^*(t)) = \begin{cases} (0,1), & \text{if} \quad 0 \le t \le \tau, \\ (0,0), & \text{if} \quad \tau < t \le T. \end{cases} \tag{6.2}$$

Now, we will continue the analysis of the optimal controls $(u_1^*(t), u_2^*(t))$. From Lemmas 8, 9 it follows that in the region $s \ge c$ the optimal trajectory $(q_*(t), s_*(t))$ corresponds to the controls $(u_1^*(t), u_2^*(t))$, which either take the values $(0,0)$, or at the moment of time $\xi \in (0,T)$ switch from the values $(1,0)$ to the values $(0,0)$, or at the moment of time $\xi \in (0,T)$

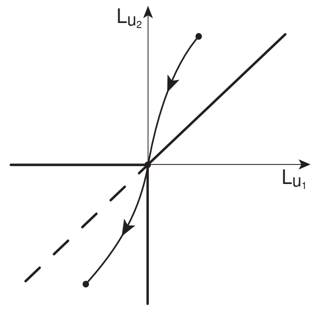


Figure 12: Behavior of the curve of switching functions $(L_{u_1}(t), L_{u_2}(t))$ when it comes to the origin.

switch from the values $(0, 1)$ to the values $(0, 0)$. Hence, for the optimal controls $(u_1^*(t), u_2^*(t))$ we have one of the types (3.10),(3.11),(6.2).

Now, we will consider the situation when at the moment of time $\theta \in (0, T)$ the optimal trajectory $(q_*(t), s_*(t))$ intersects the line $s = c$. From Lemma 2 for all $t \leq \theta$ the inequality $s_*(t) \leq c$ follows. Besides, from the arguments above and Lemmas 8, 9 we find that at the moment of time $\theta \in (0, T)$, at which the optimal trajectory $(q_*(t), s_*(t))$ intersects the line $s = c$ the corresponding controls $(u_1^*(t), u_2^*(t))$ take either the values $(0, 0)$, or the values $(1, 0)$, or the values $(0, 1)$. Further, we study each of these situations separately.

1) Let us have the equality $(u_1^*(t), u_2^*(t))|_{t=\theta} = (0, 0)$. From the relationship (3.2) the following inequalities hold:

$$L_{u_1}(\theta) < 0,\ L_{u_2}(\theta) < 0.$$

Then from the Cauchy problem (3.6) for all $t \leq \theta$ we find the inequality $L_{u_2}(t) < 0$. From Lemma 5 we see that the further behavior of the switching function $L_{u_1}(t)$ depends on the value Q_*.

If $Q_* < 0$, then we have the inequality $\dot{L}_{u_1}(t) > 0$ for all $t < \theta$. Therefore, for all $t \leq \theta$ we obtain the inequality $L_{u_1}(t) < 0$. From the relationship (3.2) for the optimal controls $(u_1^*(t), u_2^*(t))$ we find the type (3.10).

If $Q_* = 0$, then we have the equality $\dot{L}_{u_1}(t) = 0$ for all $t < \theta$. Therefore, for all $t \leq \theta$ we obtain the inequality $L_{u_1}(t) < 0$. Again, from the relationship (3.2) for the optimal controls $(u_1^*(t), u_2^*(t))$ we find the type (3.10).

If $Q_* > 0$, then we have the inequality $\dot{L}_{u_1}(t) < 0$ for all $t < \theta$. Therefore, the switching function $L_{u_1}(t)$ decreases. The behavior of this function is determined by the value $L_{u_1}(0)$. Under the condition $L_{u_1}(0) \leq 0$ we have the inequality $L_{u_1}(t) < 0$ for all $t \in (0, \theta]$. Then for the optimal controls $(u_1^*(t), u_2^*(t))$ we find the type (3.10). Under the condition $L_{u_1}(0) > 0$ we have the value $\eta \in (0, \theta)$, for which the equality $L_{u_1}(\eta) = 0$ is valid. Passing through this point the switching function $L_{u_1}(t)$ changes its sign from positive to negative. Therefore, from the relationship (3.2) we see that at the moment of time $\eta \in (0, \theta)$ the controls $(u_1^*(t), u_2^*(t))$ switch from the values $(1, 0)$ to the values $(0, 0)$. In this way, the optimal controls $(u_1^*(t), u_2^*(t))$ have the type (3.11).

2) Let us have the equality $(u_1^*(t), u_2^*(t))|_{t=\theta} = (1, 0)$. From the relationship (3.2) it follows the inequality $L_{u_1}(\theta) > 0$. Besides, the value $L_{u_2}(\theta)$ can be any real number. We will study the possible situations.

If $L_{u_2}(\theta) < 0$, then from the Cauchy problem (3.6) we find the inequality $L_{u_2}(t) < 0$ for all $t \leq \theta$. Again, we will consider the value Q_*.

If $Q_* \geq 0$, then we have the inequality $\dot{L}_{u_1}(t) \leq 0$. From the last inequality we find the inequality $L_{u_1}(t) > 0$ for all $t \in [0, \theta]$. Therefore, from the relationship (3.2) it follows that on the interval $[0, \theta)$ the controls $(u_1^*(t), u_2^*(t))$ take values $(1, 0)$. Finally, for the optimal controls $(u_1^*(t), u_2^*(t))$ we have the type (3.11).

If $Q_* < 0$, then we have the inequality $\dot{L}_{u_1}(t) > 0$. It means that the further behavior of the switching function $L_{u_1}(t)$ is determined by the value $L_{u_1}(0)$. Under the condition $L_{u_1}(0) \geq 0$ we have the inequality $L_{u_1}(t) > 0$ for all $t \in (0, \theta]$. Then for the optimal controls $(u_1^*(t), u_2^*(t))$ the analogous arguments and conclusions hold. Under the condition $L_{u_1}(0) < 0$ we have the value $\eta \in (0, \theta)$, for which the equality $L_{u_1}(\eta) = 0$ is valid. Passing through this point the switching function $L_{u_1}(t)$ changes its sign from negative to positive. Therefore, from the relationship (3.2) we see that at the moment of time $\eta \in (0, \theta)$ the controls $(u_1^*(t), u_2^*(t))$ switch from the values $(0, 0)$ to the values $(1, 0)$. In this way, the optimal controls $(u_1^*(t), u_2^*(t))$ have the type

$$(u_1^*(t), u_2^*(t)) = \begin{cases} (0, 0), & \text{if} \quad 0 \leq t \leq \eta, \\ (1, 0), & \text{if} \quad \eta < t \leq \xi, \\ (0, 0), & \text{if} \quad \xi < t \leq T. \end{cases} \tag{6.3}$$

If $L_{u_2}(\theta) \geq 0$, then from the Cauchy problem (3.6) we find the inequality $L_{u_2}(t) \geq 0$ for all $t \leq \theta$. From Lemma 11 for the switching function $L_{u_1}(t)$ we have two possibilities:

Either under the condition $L_{u_1}(0) \geq 0$ the inequality $L_{u_1}(t) > 0$ is valid for all $t \in (0, \theta]$. Therefore, from Lemma 10 we have for the controls $(u_1^*(t), u_2^*(t))$ on the interval $[0, \theta]$ the values $(1, 0)$. Then for the optimal controls $(u_1^*(t), u_2^*(t))$ we find the type (3.11).

Or under the condition $L_{u_1}(0) < 0$ we have the value $\eta \in (0, \theta)$, for which the equality $L_{u_1}(\eta) = 0$ is valid. Passing through this point the switching function $L_{u_1}(t)$ changes its sign from negative to positive. Therefore, from the relationship (3.2) we see that at the moment of time $\chi \in (\eta, \theta)$ the controls $(u_1^*(t), u_2^*(t))$ switch from the values $(0, 1)$ to the values $(1, 0)$. Finally, the optimal controls $(u_1^*(t), u_2^*(t))$ have the type

$$(u_1^*(t), u_2^*(t)) = \begin{cases} (0,1), & \text{if} \quad 0 \leq t \leq \chi, \\ (1,0), & \text{if} \quad \chi < t \leq \xi, \\ (0,0), & \text{if} \quad \xi < t \leq T. \end{cases} \tag{6.4}$$

3) Let us have the equality $(u_1^*(t), u_2^*(t))|_{t=\theta} = (0, 1)$. From the relationship (3.2) it follows the inequality $L_{u_2}(\theta) > 0$. From the Cauchy problem (3.6) we find the inequality $L_{u_2}(t) > 0$ for all $t \leq \theta$. Again, from Lemma 11 for the switching function $L_{u_1}(t)$ we have two possibilities:

Either under the condition $L_{u_1}(0) \geq 0$ the inequality $L_{u_1}(t) > 0$ is valid for all $t \in (0, \theta]$. Then from Lemma 10 we have for the optimal controls $(u_1^*(t), u_2^*(t))$ the type (6.2).

Or under the condition $L_{u_1}(0) < 0$ we have the value $\eta \in (0, \theta)$, for which the equality $L_{u_1}(\eta) = 0$ is valid. Passing through this point the switching function $L_{u_1}(t)$ changes its sign from negative to positive. Therefore, from the relationship (3.2) we see that again the optimal controls $(u_1^*(t), u_2^*(t))$ have the type (6.2).

Consequently, under the condition $l \geq 1$ the optimal controls $(u_1^*(t), u_2^*(t))$ have only one of the types (3.10),(3.11), (6.2)-(6.4).

7 Behavior of the optimal solutions at $l < 1$

Now, we will consider the situation $l < 1$. The Cases 1, 2 have been studied above. The main conclusions are contained in Lemmas 8, 9. We will investigate the Case 3. We have the following arguments. On the interval $\Delta \subset [0, T]$ under the condition $u_1^*(t) + u_2^*(t) = 1$ the controls $(u_1^*(t), u_2^*(t))$ can be expressed from the system (2.5) as follows:

$$u_1^*(t) = \frac{\dot{q}_*(t)}{\mu p q_*(t)}, \; u_2^*(t) = \frac{\dot{s}_*(t) - \beta l \dot{q}_*(t)}{\mu \beta p (l-1) q_*(t)}.$$

Next, we substitute these expressions into the functional (2.6). After necessary transformations we obtain the relationship

$$J(u_1^*, u_2^*) = \int_\Delta (Y(q_*(t), s_*(t))\dot{q}_*(t) + Z(q_*(t), s_*(t))\dot{s}_*(t))\, dt + \dots,$$

where

$$\begin{aligned} Y(q,s) &= \frac{\mu(\gamma p - p_0)q - \lambda g(s)}{\mu p q} + \frac{l}{l-1} \cdot \frac{\mu p_0 q + \lambda g(s)}{\mu p q}, \\ Z(q,s) &= -\frac{\mu p_0 q + \lambda g(s)}{\mu \beta p (l-1) q}. \end{aligned}$$

We have the contour integral at the optimal trajectory $(q_*(t), s_*(t))$ of the type

$$J(u_1^*, u_2^*) = \int_{C_*} Y(q_*, s_*) dq_* + Z(q_*, s_*) ds_* + \dots,$$

where C_* is the curve determined by this trajectory.

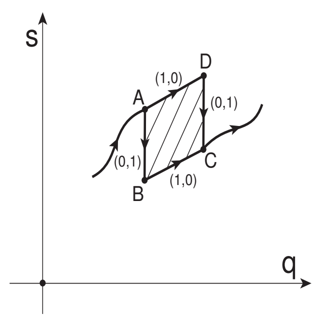


Figure 13: Optimal trajectory $(q_*(t), s_*(t))$ in Case 3 with controls $(u_1^*(t), u_2^*(t)) = (1,0)$, $(u_1^*(t), u_2^*(t)) = (0,1)$ on small interval ω.

Now, at the optimal trajectory $(q_*(t), s_*(t))$ on the small interval $\omega \subset \Delta$ we will replace the controls $(u_1^*(t), u_2^*(t))$ as shown in Figure 13. We consider the difference of corresponding values of the functionals. One of which corresponds to the optimal trajectory $(q_*(t), s_*(t))$ that contains the path ABC, and the other to the same trajectory $(q_*(t), s_*(t))$ that contains the path ADC. We have the chain of equalities:

$$J_{ABC}(u_1^*, u_2^*) - J_{ADC}(u_1^*, u_2^*) =$$

$$= \oint_{ABCDA} Y(q_*, s_*) dq_* + Z(q_*, s_*) ds_* = \iint_\Omega \frac{\lambda g(s_*) - \beta q_* \dot{g}(s_*)}{\mu \beta p (l-1) q_*^2} dq_* ds_*.$$

Here Green's Theorem was used and Ω is the region bounded by the closed path $ABCDA$. The shape of this closed path is based on the phase portraits of the system (2.5) shown in Figures 2, 3. From the analysis of the last expression, we see that in the situation when the procedure of substituting of the controls $(u_1^*(t), u_2^*(t))$ on the interval $\omega \subset \Delta$ is made in the region $s < c$, the values of the corresponding functionals are equal. So, we cannot say anything about the considered difference. In the situation, when the procedure of substituting of the controls $(u_1^*(t), u_2^*(t))$ is made in the region $s > c$, the difference of the values of the functionals is defined by the sign of the expression

$$\lambda g(s_*) - \beta q_* \dot{g}(s_*) = \frac{1}{2}\lambda(s_* - c)(s_* - 2\beta q_* - c).$$

If the inequality $s_* > 2\beta q_* + c$ is valid, then this difference is negative. If the opposite inequality $s_* < 2\beta q_* + c$ holds, then this difference is positive.

Based on these, we can support the following statement.

Lemma 12 *Let $(q_*(t), s_*(t))$ be the optimal trajectory, for which on some interval $\Delta \subset [0, T]$ the inequality $s_*(t) > c$ holds and the corresponding optimal controls $(u_1^*(t), u_2^*(t))$ obey the equality $u_1^*(t) + u_2^*(t) = 1$. Therefore, the controls $(u_1^*(t), u_2^*(t))$ on the interval Δ are of the following types:*

either take the constant value from $\{(0, 1); (1, 0)\}$,

or if the inequality $s_* > 2\beta q_* + c$ holds, they have the moment of switching $\theta \in \Delta$, for which the following equalities are valid:

$$(u_1^*(\theta - 0), u_2^*(\theta - 0)) = (1, 0),\ (u_1^*(\theta + 0), u_2^*(\theta + 0)) = (0, 1),$$

or if the inequality $s_* < 2\beta q_* + c$ holds, they have the moment of switching $\theta \in \Delta$, for which the following equalities are valid:

$$(u_1^*(\theta - 0), u_2^*(\theta - 0)) = (0, 1),\ (u_1^*(\theta + 0), u_2^*(\theta + 0)) = (1, 0).$$

Proof. For clarification, the proof of this fact we will conduct under the condition $s_* > 2\beta q_* + c$. Under the execution of the opposite condition the arguments are analogues.

We assume the contradiction. Let the following equalities hold:

$$(u_1^*(\theta - 0), u_2^*(\theta - 0)) = (0, 1),\ (u_1^*(\theta + 0), u_2^*(\theta + 0)) = (1, 0).$$

Using arguments presented above for the optimal value of functional $J_* = J(u_1^*, u_2^*)$ we find the chain of equalities:

$$J_* - J_{ADC}(u_1^*, u_2^*) = J_{ABC}(u_1^*, u_2^*) - J_{ADC}(u_1^*, u_2^*) < 0,$$

which is contradictory. Our assumption was wrong. The statement is proved. ■

Now, we will consider the situation when on the some interval $\omega \subset \Delta$ the following equality holds

$$s_*(t) = 2\beta q_*(t) + c. \tag{7.1}$$

Let us find out whether the curve determined by the equality (7.1) is a trajectory of the system (2.5) under the controls $(u_1^*(t), u_2^*(t))$. For this, we will differentiate the equality (7.1) and substitute to the obtained expression the equations of the system (2.5). We find the relationship between the controls $(u_1^*(t), u_2^*(t))$ given by

$$l - u_2^*(t) = 2u_1^*(t).$$

Taking into account the equality for these controls in the Case 3 we obtain for the controls $(u_1^*(t), u_2^*(t))$ on the interval $\omega \subset \Delta$ the formulas:

$$u_1^*(t) = l - 1,\ u_2^*(t) = 2 - l.$$

From these expressions it follows that the controls $(u_1^*(t), u_2^*(t))$, for which the equality (7.1) is valid are not admissible. Then the considered situation is impossible. From these arguments and results from [17] we have the statement.

Lemma 13 *Let there exist such moment of time $\theta \in \Delta$ that for the optimal trajectory $(q_*(t), s_*(t))$ the equality $s_*(\theta) = 2\beta q_*(\theta) + c$ is valid. Therefore, the value $\theta \in \Delta$ is not a moment of switching of the corresponding optimal controls $(u_1^*(t), u_2^*(t))$.*

Remark 5 *Lemma 12 is both valid for all points on the line $s = c$ and for the points below this line, in its small neighborhood.*

Moreover, we note that the scalar product of the derivative of the optimal trajectory $(q_*(t), s_*(t))$ and the normal vector of the line $s = 2\beta q + c$ is negative. It means that the trajectory $(q_*(t), s_*(t))$ intersects the line $s = 2\beta q + c$ only in one direction. Precisely as shown in Figure 14.

Now, we will consider the situation when the inequality $s_T \leq 2\beta q_T + c$ is valid. From Lemmas 8, 9 it follows that at the region $c \leq s \leq 2\beta q + c$ the optimal trajectory $(q_*(t), s_*(t))$ corresponds to the controls $(u_1^*(t), u_2^*(t))$, which either take the values $(0, 0)$, or at the moment of time $\xi \in (0, T)$

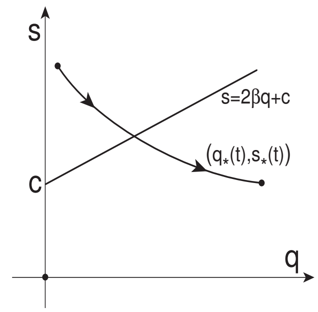


Figure 14: Behavior of the optimal trajectory $(q_*(t), s_*(t))$ when it intersects the line $s = 2\beta q + c$.

switch from the values $(1, 0)$ to the values $(0, 0)$, or at the moment of time $\xi \in (0, T)$ switch from the values $(0, 1)$ to the values $(0, 0)$. Hence, for the optimal controls $(u_1^*(t), u_2^*(t))$ we have one of the types (3.10),(3.11),(6.2).

Next, in these situations let us find out what types of the optimal controls $(u_1^*(t), u_2^*(t))$ are also possible. At first, we will study the last situation. From the analysis of trajectories of the system (2.5) (see Figure 1 - Figure 3) it follows that the optimal trajectory $(q_*(t), s_*(t))$ under the controls $(u_1^*(t), u_2^*(t)) = (0, 1)$ in the reversed time is moving up with increasing of the value $s_*(t)$. It intersects the line $s = 2\beta q + c$. And only at the region $s > 2\beta q + c$ in accordance with Lemma 12 the corresponding controls $(u_1^*(t), u_2^*(t))$ can switch at the moment of time $\chi \in (0, \xi)$ from the values $(1, 0)$ to the values $(0, 1)$. It follows from discussion after Lemma 13, that for any further decrease of the time t the controls $(u_1^*(t), u_2^*(t))$ do not have other switchings. Therefore, the optimal controls $(u_1^*(t), u_2^*(t))$ have the type

$$(u_1^*(t), u_2^*(t)) = \begin{cases} (1, 0), & \text{if} \quad 0 \leq t \leq \chi, \\ (0, 1), & \text{if} \quad \chi < t \leq \xi, \\ (0, 0), & \text{if} \quad \xi < t \leq T. \end{cases} \tag{7.2}$$

Now, we will investigate the situation when the optimal trajectory $(q_*(t), s_*(t))$ at the moment of time $\theta \in (0, T)$ intersects the line $s = c$. From Lemmas 8, 9 we find that at the moment of time $\theta \in (0, T)$ the corresponding controls $(u_1^*(t), u_2^*(t))$ take either the values $(0, 0)$, or the values $(1, 0)$. Further, we study each of these situations separately.

1) Let us have the equality $(u_1^*(t), u_2^*(t))|_{t=\theta} = (0, 0)$. From the rela-

tionship (3.2) the following inequalities hold:

$$L_{u_1}(\theta) < 0,\ L_{u_2}(\theta) < 0.$$

Using these inequalities and arguments similar to the arguments from Lemma 7, we show that for all $t \in [0, \theta]$ the following relationships hold:

$$s_*(t) \le c,\ L_{u_2}(t) = L_{u_2}(\theta) < 0. \tag{7.3}$$

Using the inequalities (7.3) and arguments similar to the arguments from the situation $l \ge 1$, we conclude that the optimal controls $(u_1^*(t), u_2^*(t))$ have one of the types (3.10),(3.11).

2) Let us have the equality $(u_1^*(t), u_2^*(t))|_{t=\theta} = (1, 0)$. From the relationship (3.2) it follows the inequality $L_{u_1}(\theta) > 0$. Besides, the value $L_{u_2}(\theta)$ can be any real number. We will study the possible situations.

If $L_{u_2}(\theta) < 0$, then executing arguments similar to the arguments from Lemma 7, we find the relationships (7.3). Using that and arguments similar to the arguments from the situation $l \ge 1$, we conclude that the optimal controls $(u_1^*(t), u_2^*(t))$ have one of the types (3.11),(6.3).

If $L_{u_2}(\theta) = 0$, then under the controls $(u_1^*(t), u_2^*(t)) = (1, 0)$ the curve of the switching functions $(L_{u_1}(t), L_{u_2}(t))$ can move in the region $s \le c$ over the horizontal axis either to the right with increasing the value L_{u_1}, or to the left with decreasing the value L_{u_1}. In the first situation from the relationship (3.2) we see that the controls $(u_1^*(t), u_2^*(t))$ keep the values $(1, 0)$. Then the optimal controls $(u_1^*(t), u_2^*(t))$ have the type (3.11). In the second situation at the moment of reaching the origin the curve of the switching functions $(L_{u_1}(t), L_{u_2}(t))$ is moving either in the region $L_{u_1} > L_{u_2}$ where the controls $(u_1^*(t), u_2^*(t))$ still take the values $(1, 0)$, or in the region $L_{u_1} < L_{u_2}$ where these controls take new values $(0, 1)$. In the first situation the optimal controls $(u_1^*(t), u_2^*(t))$ have the type (3.11). In the second situation at the moment of time $\chi \in (0, \theta)$ the switching occurs from the values $(0, 1)$ to the values $(1, 0)$. The optimal trajectory $(q_*(t), s_*(t))$ moves at the reversed time towards decreasing the value $s_*(t)$. Therefore, from the arguments presented above we conclude that the corresponding optimal controls $(u_1^*(t), u_2^*(t))$ have the type (6.4) or the type

$$(u_1^*(t), u_2^*(t)) = \begin{cases} (1,0), & \text{if} \quad 0 \le t \le \eta, \\ (0,1), & \text{if} \quad \eta < t \le \chi, \\ (1,0), & \text{if} \quad \chi < t \le \xi, \\ (0,0), & \text{if} \quad \xi < t \le T, \end{cases} \tag{7.4}$$

where $\eta \in (0, \chi)$ is the moment of switching.

If $L_{u_2}(\theta) > 0$, then executing arguments similar to the arguments presented above, we find that the optimal controls $(u_1^*(t), u_2^*(t))$ have one of the types (3.11),(6.4),(7.4).

At last, we will consider the situation when the inequality $s_T > 2\beta q_T + c$ is valid. Immediately we extract the situation, in which the optimal trajectory $(q_*(t), s_*(t))$ under the controls $(u_1^*(t), u_2^*(t)) = (0, 0)$ at the reversed time is moving down to the region $s_T \leq 2\beta q_T + c$. Then the analysis of the optimal controls $(u_1^*(t), u_2^*(t))$ is executed with arguments similar to the arguments from the previous situation.

Now, in the considered region in accordance with Lemmas 8, 9 the controls $(u_1^*(t), u_2^*(t))$ either at the moment of time $\xi \in (0, T)$ switch from the values $(1, 0)$ to the values $(0, 0)$, or at the moment of time $\xi \in (0, T)$ switch from the values $(0, 1)$ to the values $(0, 0)$. In the first situation from Lemma 12 we note that at any smaller values of time t there are not other switchings. The corresponding optimal controls $(u_1^*(t), u_2^*(t))$ have the type (3.11). In the second situation the optimal controls $(u_1^*(t), u_2^*(t))$ either have the type (6.2), or at the moment of time $\chi \in (0, \xi)$ switch from the values $(1, 0)$ to the values $(0, 1)$. In accordance with Lemma 12 the other switchings do not occur. Therefore, the optimal controls $(u_1^*(t), u_2^*(t))$ have the type (7.2).

Consequently, under the condition $l < 1$ the optimal controls $(u_1^*(t), u_2^*(t))$ have only one of the types (3.10),(3.11),(6.2)-(6.4),(7.2),(7.4).

From the analysis of the optimal control problem (2.5),(2.6) we easily find the validity of the following statement, which in some sense supplements Lemma 1.

Lemma 14 *Let $(q_*(t), s_*(t))$ be the optimal trajectory at $l < 1$. Then the following inequality holds*

$$s_*(t) \geq 0, \ t \in [0, T].$$

8 Solution of the optimal control problem

We will conduct the subsequent solution of the optimal control problem (2.5),(2.6) in the following way.

Let us introduce the set

$$S = \left\{ (\theta_1, \theta_2, \theta_3, \theta_4) \in \mathbb{R}^4 : 0 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4 \leq T \right\}.$$

Next, for any point $(\theta_1, \theta_2, \theta_3, \theta_4) \in S$ we define the controls of the type

$$(v_1^\theta(t), v_2^\theta(t)) = \begin{cases} (0,0), & \text{if} \quad 0 \le t \le \theta_1, \\ (1,0), & \text{if} \quad \theta_1 < t \le \theta_2, \\ (0,1), & \text{if} \quad \theta_2 < t \le \theta_3, \\ (1,0), & \text{if} \quad \theta_3 < t \le \theta_4, \\ (0,0), & \text{if} \quad \theta_4 < t \le T. \end{cases} \tag{8.1}$$

It is easy to see that the controls $(v_1^\theta(t), v_2^\theta(t))$ include all possible types (3.10),(3.11),(6.2)-(6.4),(7.2),(7.4) of the optimal controls $(u_1^*(t), u_2^*(t))$ at the corresponding values of switchings θ_i, $i = \overline{1,4}$.

Next, we substitute the controls $(v_1^\theta(t), v_2^\theta(t))$ into the equations of the system (2.5) and integrate it on the interval $[0,T]$. Then we substitute the corresponding formulas for the functions $q_\theta(t)$, $s_\theta(t)$ into the objective function (2.6).

Hence, we have the function of four variables

$$F(\theta_1, \theta_2, \theta_3, \theta_4) = J(v_1^\theta, v_2^\theta), \ (\theta_1, \theta_2, \theta_3, \theta_4) \in S.$$

Therefore, the optimal control problem (2.5),(2.6) can be restated as a problem of the finite dimensional optimization

$$F(\theta_1, \theta_2, \theta_3, \theta_4) \to \min_{(\theta_1, \theta_2, \theta_3, \theta_4) \in S}. \tag{8.2}$$

The methods of the numerical solution of the problem (8.2) are well developed [18, 19].

9 Conclusions

For solving the problem (2.5),(2.6) we applied the Pontryagin Maximum Principle. Our two-point boundary value problem for the Maximum Principle consists of two state and two adjoint equations. The last two of which can be replaced by the corresponding equations for the switching functions $L_{u_1}(t)$, $L_{u_2}(t)$. The presence of the term $g(s)$ does not allow this problem to be decoupled. After studying the behavior of the switching functions in depth, and using the combination of the Pontryagin Maximum Principle and Green's Theorem, we extracted three types of the optimal trajectories, respectively. Finally, we concatenated the solutions together.

We established that the optimal controls $(u_1^*(t), u_2^*(t))$ depend on the initial and final conditions and can take one of the forms given by (3.10), (3.11), (6.2)-(6.4), (7.2), (7.4).

Note that at the last time interval all possible optimal controls $(u_1^*(t), u_2^*(t))$ take values $(0,0)$, which indicates that the manufacturer does not invest in anything at that time, makes no spending, but only accumulates profit.

We introduced the controls (8.1), which include all indicated types of the optimal controls $(u_1^*(t), u_2^*(t))$ at the corresponding values of switchings. It allows us to reformulate the optimal control problem (2.5),(2.6) as a problem of the finite dimensional optimization, for which the methods of the numerical solution are well developed.

Finally, it should be noted that the ideas presented in this study can be applied to other control systems with similar properties.

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