SYMmetry ORBITS AND THEIR DATA-ANALYTIC PROPERTIES

Órbitas de simetría y sus propiedades en análisis de datos

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Abstract

The concept of data indexed by finite symmetry orbits is reviewed within the data-analytic framework of symmetry studies. Data decompositions are discussed in terms of canonical projections and Plancherel’s formulas, and interpreted in terms of orbit invariants.

Keywords: irreducible representations, irreducible characters, finite groups, canonical projections, Fourier transforms, convolution.

1 Introduction

Scalar measurements \( x : G \rightarrow \mathbb{R} \) defined on a finite group \( G \) can be naturally expressed as points

\[
x = \sum_{\tau \in G} x_{\tau} \tau
\]

in the group algebra \( \mathbb{C}G \) of \( G \), endowed with an additive (real or complex) vector space structure and a product rule

\[
(xy)_\tau = \sum_{\sigma \in G} x_{\sigma} y_{\sigma^{-1} \tau} = (x * y)_\tau
\]

given by the convolution \( x * y \) of \( x, y \in \mathbb{C}G \). A relabeling of \( x \) by \( \sigma \in G \) is given by

\[
\sigma x = \sum_{\tau} x_{\tau} \sigma \tau = \sum_{\gamma} x_{\sigma^{-1} \gamma} \gamma
\]

in which the observation \( x_{\sigma} \) indexed by \( \sigma \) replaces the observation \( x_{1} \) originally labeled by the identity. The group relabeling gives rise to the usual regular representation

\[
\phi_\sigma : (x_\tau)_{\tau \in G} \mapsto (x_{\sigma^{-1} \tau})_{\tau \in G}
\]
of $G$ into $GL_g(\mathbb{C})$, where $g$ indicates the order of $G$.

The assignment of measurements to objects, although an apparently
trivial pursuit,

\ldots when measuring some a attribute of a class of objects or
events, we associate numbers (or other familiar mathematical
entities, such as vectors) with the objects in such a way that
the properties of the attributes are faithfully represented as
numerical properties\footnote{Foundations of Measurement Volume I: Additive and Polynomial Representations, by David H. Krantz, R. Duncan Luce, Amos Tversky, and Patrick Suppes.} ...

hides in itself the implicit arbitrariness present in the association or as-
signment of numbers to objects. As a consequence, it seems desirable that
such associations be invariant, in a sense to be made precise later, to any
possible relabeling or re-assignment of measurements to labels.

To illustrate the effect of labeling in the assignments of numbers to
objects, consider the triangle with vertices \{(1, 0), (0, 2), (3, 3)\} and the
evaluation of its area

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

using Heron’s Formula, where $a$, $b$, and $c$ are the lateral lengths and

$$s = \frac{a + b + c}{2}$$

its semi-perimeter. The evaluation of the area requires the choice of one
of the possible permutations

$$\pi : \{a, b, c\} \mapsto \{\sqrt{5}, \sqrt{13}, \sqrt{10}\}$$

giving the distinct indexing of the lateral sides by the symbols \{a, b, c\}. In
this case, obviously, $\pi A = A$ for all permutations $\pi$ in the symmetric
group $S_3$, and Heron’s Formula is said to reduce symmetrically.

In contrast, the evaluation of the triangle’s index of handedness, given by

$$F = \frac{a-b}{a+b} + \frac{b-c}{b+c} + \frac{c-a}{c+a},$$

now shows, as one chooses the different permutations $\pi$ in the labeling
of the lateral sides, that the index depends on that particular choice in
accordance with

$$\pi F = \begin{cases} -F & \text{for } \pi \in \{(ab), (ac), (bc)\} \subset S_3, \\ +F & \text{for } \pi \in \{1, (abc), (acb)\} \subset S_3 \end{cases}.$$
That is, \( F \) reduces \textit{anti-symmetri
cally}.

The purpose of this short communication is presenting a brief overview
of the interpretations of orbit relabeling that are relevant to the analysis
of measurements indexed along a group orbit. The reader is referred to
[1, 2] for complete details and additional references.

\section{The canonical projections}

Indicate by \( \hat{G} \) the set of irreducible representations of a finite group \( G \), and
denote by \( \chi_\xi^\tau = \text{Tr} \xi \tau \in \mathbb{C} \) the character of \( \xi \in \hat{G} \) evaluated at \( \tau \in G \).
When considered as points in \( \mathbb{C}G \) we have \( \chi_\xi^\xi \chi_\eta^\eta = 0 \) for any distinct
non-equivalent \( \xi, \eta \in \hat{G} \), whereas \( \chi_\xi^\xi = g \chi_\xi^\xi / n_\xi \), with \( n_\xi \) denoting the
\( \mathbb{C} \)-dimension of the representation (space of) \( \xi \).

Defining
\[
\pi_\xi = \frac{n_\xi}{g} \chi_\xi \in \mathbb{C}G, \xi \in \hat{G},
\]

it then follows that
\[
\begin{align*}
\pi_\xi^2 &= \pi_\xi; & (1) \\
\pi_\xi \pi_\eta &= 0 \text{ for any two distinct } \xi, \eta \in \hat{G}; & (2) \\
\sum_{\xi \in \hat{G}} \pi_\xi &= 1 \in \mathbb{C}G. & (3)
\end{align*}
\]

This is the abstract group-algebra formulation of the canonical projections
theorem.

Given a homomorphism \( \rho \) of \( G \) into \( GL_n(\mathbb{C}) \), the linearizations
\[
\langle x, \rho \rangle = \sum_\tau x_\tau \rho_\tau, \quad x \in \mathbb{C}G
\]

are points in the enveloping (group) algebra \( A[\rho] \) of \( \{ \rho_\tau; \tau \in G \} \) such that
\[
\langle x, \rho \rangle \langle y, \rho \rangle = \langle xy, \rho \rangle \quad (4)
\]

for all \( x, y \in \mathbb{C}G \). In particular, for \( \pi_\xi \) as defined above, and \( \phi \) the regular
homomorphism of \( G \) into \( GL_g(\mathbb{C}) \), the points
\[
\mathcal{P}_\xi = \langle \bar{\pi}_\xi, \phi \rangle = \frac{n_\xi}{g} \sum_\tau \bar{\chi}_\xi^\tau \phi_\tau \in A[\phi]
\]

describe the regular canonical projections and, from (1)-(4), satisfy the
fundamental properties

1. $P^2_\xi = P_\xi$;

2. $P_\xi P_\eta = P_\eta P_\xi = 0$ for any two distinct $\xi, \eta \in \hat{G}$;

3. $\sum_{\xi \in \hat{G}} P_\xi = I$,

where $I$ indicates the $g \times g$ identity matrix. The regular projections give a class of orbit invariants based on the important fact that they commute with all elements $\phi_\tau$ of the regular representation of $G$. That is

$$\phi_\tau P_\xi = P_\xi \phi_\tau$$

for all $\tau \in G$, for all $\xi \in \hat{G}$.

As a consequence, for all $x \in \mathbb{R}^g$ the relabeling $\phi_\tau (P_\xi x)$ of $P_\xi x$ is equal to $P_\xi (\phi_\tau x)$, which remains in the projection space $W_\xi$ of $P_\xi$, for all $\tau \in G$.

For example, if $G = S_2 \cong \{1, v\}$, then

$$\phi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \phi_v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so that

$$P_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is the projection associated with the symmetric character, and

$$P_\alpha = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

is the projection associated with the anti-symmetric ($\alpha$). Moreover, $P_1 x$ defines the subspace $W_1$ generated by

$$\langle x, 1 \rangle = x_1 + x_v$$

whereas $P_\alpha x$ defines the subspace $W_\alpha$ generated by

$$\langle x, \alpha \rangle = x_1 - x_v.$$
any experiment performed on a bilateral structure such as fellow eyes or follow ears, where the notion of left-right or up-down, may be arbitrary.

The commutativity of $\phi_{\tau}$ and $P_{\xi}$ implies that

$$x'P_{\xi}x, \, \xi \in \hat{G},$$

are orbit constants. That is, the relabeling $\tau x = \phi_{\tau}x$ leads to

$$(\phi_{\tau}x)'P_{\xi}(\phi_{\tau}x) = x'(\phi_{\tau}P_{\tau}\phi_{\tau})x = x'(\phi_{\tau}^{-1}P_{\tau}\phi_{\tau})x = x'P_{\tau}x.$$

In statistical inference, these invariants are commonly referred to as variance components in the decomposition

$$\|x\|^2 = \sum_{\xi} x'P_{\xi}x, \, \xi \in \hat{G}$$

of $\|x\|^2$. This is why the canonical projections are the mechanisms behind all decompositions (or analysis) of variance for the purpose of statistically estimating and testing the magnitude of the observed orbit constants (or contrasts, in the statistical terminology).

All random samples have an intrinsic arbitrariness in the assignments of observations to their labels $\{1, 2, \ldots, n\}$, described by the permutations $\tau$ in the full symmetric group $S_n$ acting on the labels. If $\rho_{\tau}$ is the corresponding permutation matrix then we write

$$\tau x = \rho_{\tau}x, \, \tau \in S_n$$

in analogy to the regular case in which the group acts on itself. The resulting canonical projections give a decomposition

$$I = A + Q$$

of the $n \times n$ identity matrix $I$ in which $A$ is a $n \times n$ matrix with all entries equal to $1/n$ and $Q = I - A$. Moreover, $AQ = QA = 0$ and $A^2 = A, Q^2 = Q$.

The resulting orbit constants of random sampling are then the components of the decomposition

$$\|x\|^2 = x'Ax + x'Qx = n(\bar{x})^2 + \sum_{i=1}^n (x_i - \bar{x})^2,$$

namely, the sample mean and the sample variance.
Proposition 2.1. The regular projections evaluate as
\[ P_\xi x = \frac{n_\xi}{g} (\chi^\xi \ast x). \]

Proof:
\[
P_\xi x = \frac{n_\xi}{g} \sum_\tau \bar{\chi}_\tau^\xi \phi_\tau x = \frac{n_\xi}{g} \sum_\sigma \left( \sum_\tau \bar{\chi}_\tau^\xi x_{\tau^{-1}\sigma} \right) = \frac{n_\xi}{g} \sum_\sigma \left( \sum_\tau \bar{\chi}_\tau^\xi x_{\tau^{-1}\sigma} \right) \sigma = \frac{n_\xi}{g} \sum_\sigma \left( \bar{\chi}^\xi \ast x \right)_\sigma = \frac{n_\xi}{g} (\bar{\chi}^\xi \ast x).
\]

3 The orbit invariance property

Given \( x = \sum_\tau x_\tau \tau \in \mathbb{C}G \) and \( \rho \) a homomorphism of \( G \) into \( GL_n(\mathbb{C}) \), we observe that the evaluation of the linearization
\[ \langle \sigma x, \rho \rangle \]

of a relabeled point
\[ \sigma x = \sum_\gamma x_{\sigma^{-1}\gamma} \gamma \]
gives
\[ \langle \sigma x, \rho \rangle = \rho_\sigma \langle x, \rho \rangle, \]
so that when \( \rho \in \hat{G} \), the column spaces of \( \langle x, \rho \rangle \) are stable representation spaces of \( \rho \) and in that sense we say that \( \langle x, \xi \rangle \) are orbit invariants of \( x \in \mathbb{C}G \), for all \( \xi \in \hat{G} \). These linearizations, for all \( \xi \in \hat{G} \), are called the Fourier transforms of \( x \) over the finite group \( G \). In other words, The Fourier transforms over a finite group \( G \) give precisely a set of orbit invariants for that group.

The data-analytic implication is that the Fourier transforms give precisely a set of data summaries that are orbit invariant. Every instance of experimental results indexed by a (faithful) group orbit can be systematically summarized by evaluating the Fourier transforms over that group.
4 The Fourier–inverse formula

Note that

\[
\text{Tr} \left[ \xi^{-1} \langle x, \xi \rangle \right] = \text{Tr} \ \xi^{-1} \sum_{\sigma} x_{\sigma}
\]

\[
= \sum_{\sigma} x_{\sigma} \text{Tr} \ \xi^{-1} = \sum_{\sigma} x_{\sigma} \chi_{-1}^{-1}
\]

\[
= \sum_{\gamma} x_{\gamma} \chi_{\gamma}^{\xi} = \sum_{\gamma} x_{\gamma} \chi_{\gamma}^{\xi}_{-1}
\]

\[
= \sum_{\gamma} x_{\gamma} \chi_{\gamma}^{\xi} = (\chi^{\xi} * x)_{\tau}.
\]

Therefore, from Proposition 2.1,

\[
\sum_{\xi \in \hat{G}} \frac{ng_{\xi}}{n} \text{Tr} \left[ \xi^{-1} \langle x, \xi \rangle \right] = \sum_{\xi} \frac{ng_{\xi}}{n} (\chi^{\xi} * x)_{\tau}
\]

\[
= \left( \sum_{\xi} P_{\xi} x \right)_{\tau} = x_{\tau},
\]

where in the last equality we used the fact that \( \sum_{\xi \in \hat{G}} P_{\xi} = I \). This is then the Fourier–inverse formula:

\[
x_{\tau} = \sum_{\xi \in \hat{G}} \frac{ng_{\xi}}{n} \text{Tr} \left[ \xi^{-1} \langle x, \xi \rangle \right].
\]

The inverse formula is in fact the consequence of the broader result saying that the (algebra) homomorphism

\[
x \in CG \xrightarrow{\phi} \oplus_{\xi} \langle x, \xi \rangle \in \prod_{\xi} A[\xi]
\]

is an isomorphism. Indeed, if

\[
\langle x, \xi \rangle = I_{ng_{\xi}}, \text{ for all } \xi \in \hat{G}
\]

then, applying the orbit invariance property

\[
\langle \tau x, \xi \rangle = \xi_{\tau}, \text{ for all } \xi \in \hat{G}
\]

so that

\[
\frac{ng_{\xi}}{n} \langle \tau x, \xi \rangle = \frac{ng_{\xi}}{n} \xi_{\tau}
\]
and taking the trace on both sides we have

\[ \langle \tau x, \pi \xi \rangle = \frac{n\xi}{g} \chi^1 \frac{\xi}{g} \]

and summing over \( \hat{G} \) we obtain

\[ \langle \tau x, 1 \rangle = \begin{cases} 1 & \tau = 1 \\ 0 & \tau \neq 1 \end{cases} \]

that is, \( x = 1 \). In the above equality we used the fact that \( \sum_{\xi \in \hat{G}} n^2_\xi = g \) and that \( \sum_{\xi \in \hat{G}} \pi_\xi = 1 \).

### 5 Fourier basis

In general, it is possible to construct a \( g \times g \) unitary matrix \( \mathcal{F} \) such that

\[ (\mathcal{F}x)_\xi = \sqrt{\frac{n\xi}{g}} (x, \xi) \]

relative to which (basis) we have

\[ \mathcal{F} \mathcal{P}_\xi \mathcal{F}^* = \text{diag}(0, \ldots, I_{n^2_\xi}, \ldots, 0) \]

and

\[ \mathcal{F} \phi_\tau \mathcal{F}^* = \text{diag}(\ldots, I_{n_\xi} \otimes \xi_\tau, \ldots)_{\xi \in \hat{G}}. \]

To illustrate consider again the case \( G = \{1, v\} \) discussed earlier. Then,

\[ \mathcal{F} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathcal{F} \mathcal{F}^* = 1, \]

so that

\[ \mathcal{F} \mathcal{P}_1 \mathcal{F}^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \]

\[ \mathcal{F} \mathcal{P}_2 \mathcal{F}^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

Therefore, in the Fourier basis \( \{1/\sqrt{2}, \alpha/\sqrt{2}\} \) the canonical projections are the identity operators in the corresponding (irreducible) subspaces \( W_1 \oplus W_\alpha \). This, as the reader may recognize, is just a verification of Schur’s Lemma.
Moreover, if $M \in \text{Cent}\{\rho_1, \rho_\nu\}$ then it follows that $M$ must have the pattern of
\[
\begin{pmatrix}
  a & b \\
  b & a
\end{pmatrix},
\]
and, therefore, in the Fourier basis, we have
\[
\mathcal{F}M\mathcal{F}' = \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}
\]
that is, in $W_1$
\[
M = (a+b)\mathcal{P}_1
\]
and in $W_\alpha$
\[
M = (a-b)\mathcal{P}_2
\]
or
\[
M = (a+b)\mathcal{P}_1 + (a-b)\mathcal{P}_\alpha.
\]
Indeed, if $M = f_1\mathcal{P}_1 + f_\alpha\mathcal{P}_\alpha$ then $\mathcal{P}_\xi M = f_\xi\mathcal{P}_\xi$ so that, for $\xi \in \hat{G}$,
\[
f_\xi = \frac{\text{Tr} \mathcal{P}_\xi M}{\text{Tr} \mathcal{P}_\xi} = \frac{\text{Tr} \mathcal{P}_\xi M}{\text{dim} w_\xi} = \begin{cases}
  \text{Tr} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} = (a+b), \xi = 1; \\
  \text{Tr} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} = (a-b), \xi = \alpha.
\end{cases}
\]
That is: $\{\mathcal{P}_1, \mathcal{P}_\alpha\}$ is a basis for the center of the (matrix) group $\{\rho_1, \rho_\nu\}$.

In general, partitioning $G$ into its conjugacy classes $G = [\tau_1] \cup \ldots \cup [\tau_r]$ it follows that
\[
\sum_{\eta \in [\tau_i]} \eta = \sum_{\sigma} \sigma \tau_i \sigma^{-1} \in \text{Cent} \mathbb{C}G,
\]
and conversely, if $\sigma \tau \sigma^{-1} = \tau$ for all $\sigma$, then $g\sigma = \sum_{\sigma} \sigma \tau \sigma^{-1} \in \sum_{\eta \in [\tau_i]} \eta$, so that the sums
\[
\sum_{\eta \in [\tau_i]} \eta
\]
over the distinct conjugacy classes of $G$ form a basis for the Cent $\mathbb{C}G$. 

In addition, note that:

- If $M = f_1 P_1 + f_\alpha P_\alpha$ then $M^k = f_1^k P_1 + f_\alpha^k P_\alpha$, so that $e^{-iMt} = e^{-if_1^k t} P_1 + e^{-if_\alpha^k t} P_\alpha$ describes the superpositions of wave-like functions.

- If $F$ indicates the Fourier basis introduced above, and $X$ is a random variable with distribution $N(\mu, \Sigma)$ with $\Sigma$ in Cent $C G$, then the distribution of $F X$ is $N(F\mu, F\Sigma F^*)$, with
  
  $$F\Sigma F^* = \begin{bmatrix} a + b & a - b \\ a - b & a + b \end{bmatrix}, \quad F\mu = \begin{bmatrix} \mu_1 + \mu_2 \\ \mu_1 - \mu_2 \end{bmatrix}$$

  and
  
  $$\Sigma = \begin{bmatrix} \sigma^2 & \rho \sigma^2 \\ \rho \sigma^2 & \sigma^2 \end{bmatrix}, \quad a + b = \sigma^2(1 + \rho), \quad a - b = \sigma^2(1 - \rho).$$

- If $\Sigma x = \lambda x$ and $\Sigma \in$ Cent $C G$, then $\Sigma(\mathcal{P}_\xi x) = \mathcal{P}_\xi(\Sigma x) = \mathcal{P}_\xi(\lambda x) = \lambda(\mathcal{P}_\xi x)$, that is, all $\mathcal{P}_\xi x$ are eigenvectors associated with the eigenvalue $\lambda$.

- Symmetric tensors. Define $\Pi_f(A) = A \otimes \ldots \otimes A$. Take $M = \lambda_1 P_1 + \lambda_\alpha P_\alpha \in$ Cent $C G$. Then $\Pi_f(M)$ is a symmetric tensor [3] in the sense that it commutes with $\Pi_f(\rho\tau)$ for all $\tau \in G$. That is, the commutator
  
  $$[\Pi_f(\rho\tau), \Pi_f(M)] = 0, \forall \tau \in G.$$  

In quantum mechanics, the symmetric tensors are the observables of the system, whereas

$$\Pi_f(M)x$$

are the possible states of the system.

To illustrate, let $f = 2$, so that

$$\Pi_2(M) = \lambda_1^2 P_1 \otimes P_1 + \lambda_1 \lambda_2 P_1 \otimes P_\alpha + \lambda_2 \lambda_1 P_\alpha \otimes P_1 + \lambda_\alpha^2 P_\alpha \otimes P_\alpha$$

is a symmetric tensor. Take any $x$ as a superposition of

$$\{\chi_1 \otimes \chi_1, \chi_1 \otimes \chi_\alpha, \chi_\alpha \otimes \chi_1, \chi_\alpha \otimes \chi_\alpha\}.$$
Then,
\[ \Pi_2(M)(\chi_1 \otimes \chi_1) = \lambda_1^2 \chi_1 \otimes \chi_2, \]
\[ \Pi_2(M)(\chi_1 \otimes \chi_\alpha) = \lambda_1 \lambda_2 \chi_1 \otimes \chi_\alpha, \]
\[ \Pi_2(M)(\chi_\alpha \otimes \chi_1) = \lambda_2 \lambda_1 \chi_\alpha \otimes \chi_1, \]
\[ \Pi_2(M)(\chi_\alpha \otimes \chi_\alpha) = \lambda_2^2 \chi_\alpha \otimes \chi_\alpha. \]

Note that the two states \( \chi_1 \otimes \chi_\alpha \) and \( \chi_\alpha \otimes \chi_1 \) share the same (energy) level \( \lambda_1 \lambda_2 \).

6 Parseval’s and Plancherel’s equalities

From the Fourier basis definition it follows that
\[ \|x\|^2 = x^* x = (F_x)^*(F_x) = \sum_{\xi \in \hat{G}} \frac{n_\xi}{g} \| \langle x, \xi \rangle \|^2, \]
which is the Parseval’s equality. Similarly we obtain Plancherel’s equality
\[ x^* y = \sum_{\xi \in \hat{G}} \frac{n_\xi}{g} \| \langle x, \xi \rangle \| \| \langle y, \xi \rangle \|. \]

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References

